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On the relative coexistence of fixed points and period-two solutions near border-collision bifurcations

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ABSTRACT

At a border-collision bifurcation a fixed point of a piecewise-smooth map intersects a surface where the functional form of the map changes. Near a generic border-collision bifurcation there are two fixed points, each of which exists on one side of the bifurcation. A simple eigenvalue condition indicates whether the fixed points exist on different sides of the bifurcation (this case can be interpreted as the persistence of a single fixed point), or on the same side of the bifurcation (in which case the bifurcation is akin to a saddle–node bifurcation). A similar eigenvalue condition indicates whether or not there exists a period-two solution on one side of the bifurcation. Previously these conditions have been combined to obtain five distinct scenarios for the existence and relative coexistence of fixed points and period-two solutions near border-collision bifurcations. In this Letter it is shown that one of these scenarios, namely that two fixed points exist on one side of the bifurcation and a period-two solution exists on the other side of the bifurcation, cannot occur. The remaining four scenarios are feasible. Therefore there are exactly four distinct scenarios for fixed points and period-two solutions near border-collision bifurcations.

1. Introduction

A piecewise-smooth map on $\mathcal{M} \subset \mathbb{R}^N$ is a discrete-time dynamical system

$$X_{i+1} = F^j(X_i), \quad X_i \in \mathcal{M}_j, \tag{1.1}$$

where the regions $\mathcal{M}_j$ form a partition of the domain $\mathcal{M}$, and each $F^j : \mathcal{M}_j \to \mathcal{M}$ is a smooth function. The boundaries of the $\mathcal{M}_j$, termed switching manifolds, are assumed to be either smooth or piecewise-smooth surfaces. Piecewise-smooth maps are used to model oscillatory dynamics in systems involving abrupt events, such as mechanical systems with impacts [1], power electronics with switching events [2], and economics systems with non-negativity conditions or optimisation [3].

As parameters are varied, a fixed point of (1.1) may collide with a switching manifold. If, near the collision, the switching manifold is smooth, (1.1) is continuous, and the derivatives $D_X F^j$ are bounded, then the intersection is known as a *border-collision bifurcation*. For more general scenarios the reader is referred to [4]. Dynamics near a border-collision bifurcation of (1.1) is well-approximated by a piecewise-linear, continuous map, which can be put in the form

$$x_{i+1} = \begin{cases} A_L x_i + b\mu, & s_i \le 0 \\ A_R x_i + b\mu, & s_i \ge 0, \end{cases} \tag{1.2}$$

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where, throughout this Letter, $s = e_1^{\mathsf{T}} x$ denotes the first component of $x \in \mathbb{R}^N$. In (1.2), A_L and A_R are real-valued $N \times N$ matrices, $b \in \mathbb{R}^N$, and $\mu \in \mathbb{R}$ is the primary bifurcation parameter: the border-collision bifurcation occurs at $x = 0$ when $\mu = 0$. The requirement that (1.2) is continuous implies

$$A_R = A_L + \xi e_1^{\mathsf{T}}, \tag{1.3}$$

for some $\xi \in \mathbb{R}^N$.

A fixed point of (1.2) must be a fixed point of one of the two half-maps of (1.2):

$$f^L(x_i) = A_L x_i + b\mu, \qquad f^R(x_i) = A_R x_i + b\mu. \tag{1.4}$$

As long as 1 is not an eigenvalue of A_L and A_R, f^L and f^R have unique fixed points,

$$x^L = (I - A_L)^{-1} b\mu, \qquad x^R = (I - A_R)^{-1} b\mu. \tag{1.5}$$

The point x^L is a fixed point of (1.2), and said to be admissible, if $s^L \leq 0$. Similarly, x^R is admissible if $s^R \geq 0$. Since x^L and x^R are linear functions of μ, generically x^L and x^R are each admissible for exactly one sign of μ. In general, for the purposes of characterising the behaviour of (1.2), it suffices to consider only the sign of μ, because the structure of the dynamics of (1.2) is independent of the magnitude of μ.

Other invariant sets may be created in border-collision bifurcations, such as periodic solutions, invariant circles, and chaotic sets [4–9], as well as exotic dynamics such as multi-dimensional attractors [10], and infinitely many coexisting attractors [11]. This Letter concerns only fixed points and period-two solutions. Period-two solutions were first explored by Mark Feigin in the 1970s [12,13], and were described more recently in [4,14]. The creation of a period-two solution in a border-collision bifurcation has different scaling properties than a period-doubling bifurcation, and such differences can have important physical interpretations [15].

In generic situations, (1.2) either has no period-two solution for either sign of μ, or has an *LR*-cycle (a period-two solution consisting of one point on each side of $s = 0$) for exactly one sign of μ [12]. In [13], Feigin showed that the relative coexistence of the fixed points x^L and x^R is determined by a simple condition on the eigenvalues of A_L and A_R, and that a similar condition indicates whether or not an *LR*-cycle exists for one sign of μ. This is one of the most far-reaching results in the bifurcation theory of nonsmooth dynamical systems, because it applies to maps of any number of dimensions. Centre manifold analysis, which is the key tool for dimension reduction, requires local differentiability and so usually cannot be applied to bifurcations specific to nonsmooth dynamical systems, such as border-collision bifurcations [16].

By directly combining the two generic cases for the nature of both fixed points and period-two solutions, it appears that border-collision bifurcations can be categorised into five basic scenarios. In the absence of an *LR*-cycle there are two scenarios: either x^L and x^R are admissible for different signs of μ, Fig. 1(A), or x^L and x^R are admissible for the same sign of μ, Fig. 1(B). If there exists an *LR*-cycle, and x^L and x^R are admissible for different signs of μ, then, trivially, the *LR*-cycle coexists with exactly one fixed point, Fig. 1(C). Finally, if there exists an *LR*-cycle, and x^L and x^R are admissible for the same sign of μ, it appears that there are two scenarios. The *LR*-cycle could either coexist with x^L and x^R, as in Fig. 1(D), or coexist with neither x^L or x^R. In [13], Feigin noted that the latter scenario is not possible in one-dimension ($N = 1$) in view of Sharkovskii's theorem [17]. Feigin further stated that this scenario is not possible for $N = 2$ (but did not provide a proof), and conjectured that the scenario is not possible for any $N \in \mathbb{Z}^+$. The purpose of this Letter is to prove this conjecture.

Each of the four scenarios of Fig. 1 is possible for (1.2) in any number of dimensions. In Fig. 1 the scenarios are illustrated for (1.2) with $N = 1$, for which (1.2) is written as

$$x_{i+1} = \begin{cases} a_L x_i + \mu, & x_i \leq 0 \\ a_R x_i + \mu, & x_i \geq 0, \end{cases} \tag{1.6}$$

where $a_L, a_R \in \mathbb{R}$.

The remainder of this Letter is organised as follows. Calculations for fixed points and period-two solutions of (1.2) are given in Section 2 and Section 3, respectively. The basic border-collision bifurcation scenarios formed by considering all generic possibilities for fixed points and period-two solutions are described in Section 4. In Section 5 it is proved that a non-degenerate period-two solution of (1.2) must coexist with a fixed point. Finally, Section 6 presents a brief summary and outlook.

2. Fixed points

In order to compare the values of s^L and s^R (the first components of x^L and x^R (1.5)), we let

$$\varrho^{\mathsf{T}} = e_1^{\mathsf{T}} \mathrm{adj}(I - A_L), \tag{2.1}$$

where $\mathrm{adj}(A)$ denotes the *adjugate* of a square matrix A. Recall, if A is nonsingular, then $A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$. Thus, by (1.5) we have

$$s^L = \frac{\varrho^{\mathsf{T}} b}{\det(I - A_L)} \mu. \tag{2.2}$$