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No arbitrage in a simple credit risk model

Marek Capiński [a,*], Tomasz Zastawniak [b]

[a] Faculty of Mathematics, AGH-University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland

[b] Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

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ABSTRACT

In a simple credit risk model we find an equivalent condition to the no-simple-arbitrage principle and show that it is insufficient for an equivalent martingale measure to exist.

1. Introduction

The assumption that there is no arbitrage (intuitively, a profit with no risk and no investment is impossible) is the basis of pricing methodology in mathematical finance. By the first fundamental theorem [1], a condition stronger than no arbitrage, called no free lunch with vanishing risk, holds if and only if there exists an equivalent martingale measure. In the credit risk literature (for example [2] and references therein) it is assumed that the model is free of arbitrage in the sense that an equivalent martingale measure exists. But, as we shall see, this is not so clear-cut. For the simplest possible case we establish an equivalent condition to the no-simple-arbitrage principle, and then give an example showing that in fact an equivalent martingale measure may not exist. We also discuss the possibility of finding a strategy which gives a version of arbitrage (free lunch with bounded risk).

2. No-simple-arbitrage principle

We consider a market consisting of two zero-coupon bonds, a non-defaultable bond with prices $B(t,T) = e^{-r(T-t)}$, where $r > 0$ is a constant, and defaultable bond with prices denoted by $D(t,T)$, for $0 \le t \le T$. Default is triggered by a random time $\tau > 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$, where P is the physical probability. By $(\mathcal{I}_t)_{t\ge 0}$ we denote the filtration generated by the default indicator process $I(t) = \mathbf{1}_{\{\tau \le t\}}$. The events $\{\tau > T\}$ and $\{s < \tau \le t\}$ for each $s, t \in [0,T]$ such that $s < t$ are assumed to have positive probability. We assume that $D(t,T)$ is a stochastic process adapted to $(\mathcal{I}_t)_{t\ge 0}$, with right-continuous paths, and such that $D(T,T) = \mathbf{1}_{\{\tau > T\}}$ (so-called zero recovery).

We need the following property of $(\mathcal{I}_t)_{t\ge 0}$-stopping times; see [3,4].

Lemma 1. *For any $(\mathcal{I}_t)_{t\ge 0}$-stopping time σ satisfying $0 \le \sigma \le T$, there exists a deterministic constant $s \in [0,T]$ such that $\sigma \ge \min(\tau, s)$, $\sigma = s$ on $\{s < \tau\}$, and $\{\sigma < \tau\} = \{s < \tau\}$ is an atom in the sigma-field $\mathcal{I}_\sigma$.*

By a *simple strategy* we mean a pair of processes $\varphi(t) = (\varphi_B(t), \varphi_D(t))$ representing positions in $B(t,T)$ and $D(t,T)$ for $t \in [0,T]$ such that there are $(\mathcal{I}_t)_{t\ge 0}$-stopping times $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = T$ and $\mathcal{I}_{\sigma_n}$-measurable random variables

* Corresponding author. Tel.: +48 122675919.

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$X_n, Y_n, n = 0, 1, \ldots, N-1$ such that $\varphi_B(t) = X_n, \varphi_D(t) = Y_n$ for $t \in (\sigma_n, \sigma_{n+1}]$ and $\varphi_B(0) = X_0, \varphi_D(0) = Y_0$. The time *t value* of such a strategy is $V_\varphi(t) = \varphi_B(t)B(t,T) + \varphi_D(t)D(t,T)$. The strategy is *self-financing* whenever for each $n = 0, 1, \ldots, N-1$ we have $V_\varphi(\sigma_n) = \lim_{t\downarrow\sigma_n} V_\varphi(t)$ a.s. A *simple arbitrage strategy* is defined as a self-financing simple strategy φ such that $V_\varphi(0) = 0, V_\varphi(T) \geq 0$ a.s., and $V_\varphi(T) > 0$ with positive probability. If such a strategy is impossible, we say that the *no-simple-arbitrage* (NSA) principle holds.

Theorem 2. *The NSA principle holds if and only if there is a strictly increasing right-continuous function $\Gamma : [0, T] \to \mathbb{R}$ with $\Gamma(0) = 0$ such that*

$$D(t,T) = e^{-r(T-t)-(\Gamma(T)-\Gamma(t))}\mathbf{1}_{\{\tau>t\}}, \quad t \in [0,T]. \tag{1}$$

Proof. Suppose that the NSA principle holds. Since $D(t,T)$ is $\mathcal{I}_t$-measurable, it can be represented as

$$D(t,T) = \eta_t(\tau)\mathbf{1}_{\{\tau\leq t\}} + c_t\mathbf{1}_{\{\tau>t\}},$$

where $\eta_t : \mathbb{R} \to \mathbb{R}$ is a Borel function and $c_t \in \mathbb{R}$ is deterministic ([5], p. 124).

We claim that $\eta_t(\tau)\mathbf{1}_{\{\tau\leq t\}} = 0$ a.s. and $c_t > 0$. For $t = T$ the claim is true because $D(T,T) = \mathbf{1}_{\{\tau>T\}}$. Now take any $t \in [0,T)$. Let $A = \left\{\eta_t(\tau)\mathbf{1}_{\{\tau\leq t\}} > 0\right\}$ and put $\varphi_B(u) = \varphi_D(u) = 0$ for all $u \in [0,t]$, and $\varphi_B(u) = \frac{\eta_t(\tau)}{B(t,T)}\mathbf{1}_A$, $\varphi_D(u) = -\mathbf{1}_A$ for all $u \in (t,T]$. This simple strategy is self-financing given that $D(t,T)$ is assumed to be right-continuous. The initial value is $V_\varphi(0) = 0$ and the final value is $V_\varphi(T) = \frac{\eta_t(\tau)}{B(t,T)}\mathbf{1}_A$, strictly positive on A and 0 otherwise; hence $P(A) = 0$ by NSA. Now let $B = \left\{\eta_t(\tau)\mathbf{1}_{\{\tau\leq t\}} < 0\right\}$ and put $\varphi_B(u) = \varphi_D(u) = 0$ for all $u \in [0,t]$, and $\varphi_B(u) = -\frac{\eta_t(\tau)}{B(t,T)}\mathbf{1}_B$, $\varphi_D(u) = \mathbf{1}_B$ for all $u \in (t,T]$. This is also a self-financing simple strategy, with $V_\varphi(0) = 0$ and $V_\varphi(T) = -\frac{\eta_t(\tau)}{B(t,T)}\mathbf{1}_B$ strictly positive on B and 0 otherwise. Once again, $P(B) = 0$ by NSA. As a result, $D(t,T) = c_t\mathbf{1}_{\{\tau>t\}}$. If $c_t \leq 0$ for some $t \in [0,T)$, we take $\varphi_B(u) = \varphi_D(u) = 0$ for all $u \in [0,t]$, and $\varphi_D(u) = \mathbf{1}_{\{\tau>t\}}, \varphi_B(u) = -\frac{c_t}{B(t,T)}\mathbf{1}_{\{\tau>t\}}$ for all $u \in (t,T]$, which is a self-financing simple strategy. Since $V_\varphi(0) = 0$, $V_\varphi(T) = -\frac{c_t}{B(t,T)}\mathbf{1}_{\{\tau>t\}} + \mathbf{1}_{\{\tau>T\}} \geq 0$ and $V_\varphi(T) > 0$ on $\{\tau > T\}$, we have a simple arbitrage strategy, contradicting the NSA principle. The claim has been verified.

Since $D(t,T)$ has right-continuous paths and $c_t > 0$ with $c_T = 1$,

$$D(t,T) = c_t\mathbf{1}_{\{\tau>t\}} = e^{-r(T-t)-(\Gamma(T)-\Gamma(t))}\mathbf{1}_{\{\tau>t\}}$$

for some right-continuous function $\Gamma : [0,T] \to \mathbb{R}$ such that $\Gamma(0) = 0$.

It remains to show that Γ is strictly increasing. Take $t_1 < t_2$. Put $\varphi_D(u) = \varphi_B(u) = 0$ for all $u \in [0,t_1]$, $\varphi_B(u) = \mathbf{1}_{\{\tau>t_1\}}$, $\varphi_D(u) = -e^{\Gamma(T)-\Gamma(t_1)}\mathbf{1}_{\{\tau>t_1\}}$ for all $u \in (t_1,t_2]$, and $\varphi_B(u) = \mathbf{1}_{\{\tau>t_1\}}\left(1 - e^{\Gamma(t_2)-\Gamma(t_1)}\mathbf{1}_{\{\tau>t_2\}}\right)$, $\varphi_D(u) = 0$ for all $u \in (t_2,T]$. This is a simple strategy, which is self-financing given that $D(t,T)$ is right-continuous. The initial value of the strategy is $V_\varphi(0) = 0$ and the final value is

$$V_\varphi(T) = \mathbf{1}_{\{t_1<\tau\leq t_2\}} + \left(1 - e^{\Gamma(t_2)-\Gamma(t_1)}\right)\mathbf{1}_{\{t_2<\tau\}}.$$

Since both events $\{t_2 < \tau\}$ and $\{t_1 < \tau \leq t_2\}$ are assumed to have positive probability, it follows that $1 < e^{\Gamma(t_2)-\Gamma(t_1)}$ or else the NSA principle would be violated. It means that $\Gamma(t_1) < \Gamma(t_2)$.

Now suppose that $D(t,T)$ can be represented as (1) for some strictly increasing right-continuous function $\Gamma : [0,T] \to \mathbb{R}$ with $\Gamma(0) = 0$. We want to show that the NSA principle holds. Let $\varphi = (\varphi_B, \varphi_D)$ be a simple strategy with rebalancing times σ_n and values X_n, Y_n. For each n, since σ_n is an $(\mathcal{I}_t)_{t\geq0}$-stopping time such that $0 \leq \sigma_n \leq T$, we have the corresponding deterministic time $s_n \in [0,T]$ from Lemma 1 such that $\sigma_n = s_n$ on $\{\sigma_n < \tau\} = \{s_n < \tau\}$, which is an atom in $\mathcal{I}_{\sigma_n}$. Suppose that $V_\varphi(0) = 0$ and $V_\varphi(T) \geq 0$ a.s. We shall show that $V_\varphi(T) = 0$ a.s., so there is no simple arbitrage strategy. We consider two cases, which exhaust all possibilities: (a) $V_\varphi(\sigma_n) \geq 0$ a.s. for each $n = 0, 1, \ldots, N$, or (b) $V_\varphi(\sigma_n) < 0$ with positive probability for some $n = 1, \ldots, N-1$.

In case (a) we can show by induction that $V_\varphi(\sigma_n) = 0$ a.s. for each $n = 0, 1, \ldots, N$. Indeed, we have $V_\varphi(\sigma_0) = V_\varphi(0) = 0$. Next, suppose that $V_\varphi(\sigma_n) = 0$ a.s. for some $n = 0, 1, \ldots, N-1$. Self-financing at σ_n gives

$$\begin{aligned} 0 = V_\varphi(\sigma_n) &= \lim_{t\downarrow\sigma_n} V_\varphi(t) = \lim_{t\downarrow\sigma_n}(X_nB(t,T) + Y_nD(t,T)) \\ &= X_ne^{-r(T-\sigma_n)} + Y_ne^{-r(T-\sigma_n)-(\Gamma(T)-\Gamma(\sigma_n))}\mathbf{1}_{\{\sigma_n<\tau\}} \quad \text{a.s.} \end{aligned}$$

Hence $X_n = -Y_ne^{-(\Gamma(T)-\Gamma(\sigma_n))}\mathbf{1}_{\{\sigma_n<\tau\}}$ a.s. It follows that

$$\begin{aligned} V_\varphi(\sigma_{n+1}) &= X_nB(\sigma_{n+1},T) + Y_nD(\sigma_{n+1},T) \\ &= -Y_ne^{-r\left(T-\sigma_{n+1}\right)}e^{-(\Gamma(T)-\Gamma(\sigma_n))}\mathbf{1}_{\{\sigma_n<\tau\leq\sigma_{n+1}\}} \\ &\quad + Y_ne^{-r\left(T-\sigma_{n+1}\right)}e^{-\Gamma(T)}\left(e^{\Gamma(\sigma_{n+1})} - e^{\Gamma(\sigma_n)}\right)\mathbf{1}_{\{\sigma_{n+1}<\tau\}} \quad \text{a.s.} \end{aligned}$$

Since Y_n is $\mathcal{I}_{\sigma_n}$-measurable and $\{\sigma_n < \tau\}$ is an atom in $\mathcal{I}_{\sigma_n}$, it follows that Y_n is constant on $\{\sigma_n < \tau\}$. Suppose that Y_n is a negative constant on $\{\sigma_n < \tau\} \supset \{\sigma_{n+1} < \tau\}$. Since Γ is strictly increasing, we have $e^{\Gamma(\sigma_{n+1})} > e^{\Gamma(\sigma_n)}$, so $V_\varphi(\sigma_{n+1}) < 0$ on