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The residual symmetry of the $(2+1)$-dimensional coupled Burgers equation

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ABSTRACT

By means of the standard truncated Painlevé expansion, we derive the residual symmetry of the $(2+1)$-dimensional coupled Burgers equation. This kind of the residual symmetry is localized in the properly prolonged system with the Lie point symmetry vector. Based on these obtained symmetries, some different transformation invariances are derived. Furthermore, the reduction solution (especially the interactive solution) is obtained with the help of a generalized tanh-function expansion approach.

1. Introduction

As we all know, since the Lie group theory was introduced by Sophus Lie to study differential equations [1], the symmetry theory has been an important subject in studying nonlinear equations. Utilizing either the classical, non-classical Lie group methods [2,3] or the direct reduction approach [4–9], scientists can try to reduce the dimensions of partial differential equations and derive some exact solutions. However, for an integrable system, there may exist infinite nonlocal symmetries which are linked to Bäcklund transformation (BT), Darboux transformation (DT), conformal invariance and negative hierarchies and so on.

Recently, Lou et al. [10,11] developed the so-called the residual symmetry, and proposed that the symmetry related to the Painlevé truncated expansion is just the residue with respect to the singular manifold. Considering the localizations of the residual symmetries, many types of nonlocal symmetries can be localized to Lie point symmetries by introducing suitable prolonged systems [12–17]. At the same time, by means of the repeated symmetry reduction approach, many infinite symmetries and exact solutions of the Burgers equation were derived by Lou [18].

In this paper, we use the residual related symmetry reduction and the generalized tanh-function expansion approach to study the $(2+1)$-dimensional coupled Burgers equation, which is a fundamental partial differential equation in the field of fluid mechanics, gas dynamics and traffic flow.

2. The residual symmetry of the $(2+1)$-dimensional coupled Burgers' equation

The $(2+1)$-dimensional coupled Burgers equation can be written as

$$u_t - u_{xx} - u_{yy} - 2uu_x - 2vu_y = 0, \qquad v_t - v_{xx} - v_{yy} - 2uv_x - 2vv_y = 0. \tag{1}$$

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By introducing a special spectral problem with q dependent variables $u_0, u_1, \ldots, u_{q-1}$, Ma proposed the coupled Burgers system which is a typical integrable hierarchy [19]. By means of the variable separation approach, authors [20] obtained several new multisoliton excitations of the coupled Burgers equation (1) and revealed some novel features or interesting behaviors of these structures. Moreover, via the modified direct method, the finite symmetry transformation group of the (2 + 1)-dimensional coupled Burgers equation was also studied [21].

The above (2 + 1)-dimensional coupled Burgers equation is Painlevé integrable, and its truncated expansion can be expressed as

$$u = \sum_{i=0}^{a} u_i f^{i-a}, \qquad v = \sum_{j=0}^{b} u_j f^{j-b}, \tag{2}$$

where a and b are two positive integers, and f is a singular manifold.

Substituting Eq. (2) into Eq. (1), balancing the nonlinear term and the dispersion term, we can obtain $a = b = 1$. That is, the truncated Painlevé expansion is

$$u = \frac{u_0}{f} + u_1, \qquad v = \frac{v_0}{f} + v_1. \tag{3}$$

We can prove that the residuals u_0 and v_0 are the symmetries corresponding to the solutions u_1 and v_1.

Substituting Eq. (3) into Eq. (1), we have

$$\begin{aligned} &u_{1,t} - u_{1,xx} - u_{1,yy} - 2u_1u_{1,x} - 2v_1u_{1,y} + (u_{0,t} - u_{0,xx} - u_{0,yy} - 2(u_0u_1)_x - 2v_0u_{1,y} - 2v_1u_{0,y})f^{-1} \\ &\quad + (u_0(f_{xx} - f_t + 2v_1f_y + f_{yy} - 2u_{0,x} + 2u_1f_x) + 2u_{0,x}f_x + 2u_{0,y}f_y - 2v_0u_{0,y})f^{-2} \\ &\quad + 2u_0(u_0f_x - f_x^2 - f_y^2 + v_0f_y)f^{-3} = 0, \end{aligned} \tag{4}$$

and

$$\begin{aligned} &v_{1,t} - v_{1,xx} - v_{1,yy} - 2u_1v_{1,x} - 2v_1v_{1,y} + (v_{0,t} - v_{0,xx} - v_{0,yy} - 2(v_0v_1)_y - 2u_0v_{1,x} - 2u_1v_{0,x})f^{-1} \\ &\quad + (v_0(f_{xx} - f_t + 2u_1f_x + f_{yy} - 2v_{0,y} + 2v_1f_y) + 2v_{0,x}f_x + 2v_{0,y}f_y - 2u_0v_{0,x})f^{-2} \\ &\quad + 2v_0(u_0f_x - f_x^2 - f_y^2 + v_0f_y)f^{-3} = 0. \end{aligned} \tag{5}$$

Vanishing the coefficients of f^{-3} in Eqs. (4) and (5), we obtain

$$u_0 = f_x, \qquad v_0 = f_y. \tag{6}$$

Substituting Eq. (6) into Eqs. (4) and (5), we can derive the coefficients of f^{-2} and f^{-1} as follows:

$$f_t = f_{xx} + f_{yy} + 2u_1f_x + 2v_1f_y,\ v_{1,x} = u_{1,y}, \tag{7}$$

and u_1 and v_1 in solution (5) satisfy

$$u_{1,t} - u_{1,xx} - u_{1,yy} - 2u_1u_{1,x} - 2v_1u_{1,y} = 0, \tag{8}$$

and

$$v_{1,t} - v_{1,xx} - v_{1,yy} - 2u_1v_{1,x} - 2v_1v_{1,y} = 0. \tag{9}$$

Therefore, the standard truncated Painlevé expansion

$$u = \frac{f_x}{f} + u_1, \qquad v = \frac{f_y}{f} + v_1 \tag{10}$$

is a Bäcklund transformation (BT) between $\{u, v\}$ and $\{u_1, v_1\}$ if the latter satisfies Eq. (7).

Next, we introduce the variables

$$g \equiv f_x, \qquad h \equiv f_y, \tag{11}$$

which mean

$$h_x = g_y, \tag{12}$$

$$f_t = g_x + 2v_1h + h_y + 2gu_1, \tag{13}$$

$$g_t = g_{xx} + 2v_{1,x}h + 2v_1g_y + g_{yy} + 2g_xu_1 + 2gu_{1,x}, \tag{14}$$

$$h_t = g_{xy} + 2v_{1,y}h + 2v_1h_y + h_{yy} + 2g_yu_1 + 2gu_{1,y}. \tag{15}$$