Applied Mathematics Letters 37 (2014) 86–90

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Discrete-space partial dynamic equations on time scales and applications to stochastic processes

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ARTICLE INFO

Article history:
Received 18 May 2014
Received in revised form 8 June 2014
Accepted 8 June 2014
Available online 14 June 2014

Keywords:
Time scales
Partial dynamic equations
Stochastic process
Nonhomogeneous Markov process
Continuous dependence

ABSTRACT

We consider a general class of discrete-space linear partial dynamic equations. The basic properties of solutions are provided (existence and uniqueness, sign preservation, maximum principle). Above all, we derive the following main results: first, we prove that the solutions depend continuously on the choice of the time scale. Second, we show that, under certain conditions, the solutions describe probability distributions of nonhomogeneous Markov processes, and that their time integrals remain the same for all underlying regular time scales.

1. Introduction

Time scales calculus was created in order to study phenomena in discrete and continuous dynamical systems under one roof [1,2]. It has proved to be useful not only in various theoretical considerations (see e.g. [3,4]), but also in the study of applied problems in areas where discrete and continuous-time models naturally coexist, such as economics [5,6] or control theory [7]. On the other hand, one mathematical field where continuous and discrete approaches are in balance has been almost absent—probability theory and stochastic processes. Only recently, stochastic dynamic equations have been studied in [8]. Additionally, our recent papers [9–11] were devoted to discrete-space partial dynamic equations and their relation to discrete-state Markov processes. In particular, Poisson–Bernoulli processes are related to solutions of the discrete-space dynamic transport equation, and random walks correspond to discrete-space dynamic diffusion equations. Consequently, the investigation of partial dynamic equations enables us to study the properties of these processes.

In this paper, we explore discrete-space partial dynamic equations and their connection to discrete-state stochastic processes in a more general context. We focus on the linear partial dynamic equation

$$u^{\Delta_t}(x,t) = \sum_{i=-m}^{m} a_i u(x+i,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}, \tag{1.1}$$

where $m \in \mathbb{N}$, $a_{-m}, \dots, a_m \in \mathbb{R}$, and $\mathbb{T}$ is a time scale (a closed subset of $\mathbb{R}$). The symbol u^{Δ_t} denotes the partial Δ-derivative with respect to t, which coincides with the standard partial derivative u_t when $\mathbb{T} = \mathbb{R}$, or with the forward partial difference $\Delta_t u$ when $\mathbb{T} = \mathbb{Z}$. Since the differences with respect to x are not used, we omit the lower index t in u^{Δ_t} and write u^{Δ} only. Throughout the paper, we assume that $\mathbb{T}$ is a closed subset of $[0,\infty)$ and $0 \in \mathbb{T}$. The time scale intervals are denoted by $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$.

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http://dx.doi.org/10.1016/j.aml.2014.06.002
0893-9659/

Equations related to (1.1) have been studied in two directions. First, lattice dynamical systems with continuous or discrete time were discussed by several authors, see e.g. [12,13]. Second, partial dynamic equations on general time and space structures have been considered in [14,15].

In Section 2, we extend the results from [11] and discuss some basic properties of solutions to (1.1), such as existence and uniqueness, sum and sign preservation, and maximum principle. In Section 3, we prove a general theorem concerning the continuous dependence of solutions to (1.1) on initial values, coefficients on the right-hand side, as well as the choice of time scale. In Section 4, we show that under certain conditions, solutions of (1.1) correspond to probability distributions of discrete-state Markov processes. Using this probabilistic interpretation, we prove that the time integrals $\int_0^\infty u(x,t)\Delta t$, which give the expected value of the total time spent by the process in state x, do not depend on the choice of the time scale. Finally, we present examples illustrating the relation between (1.1) and Markov processes.

2. Basic results

Throughout the paper, we consider the initial value problem

$$\begin{cases} u^{\Delta}(x,t)=\displaystyle\sum_{i=-m}^{m} a_i u(x+i,t), & x\in\mathbb{Z},\ t\in\mathbb{T},\\ u(x,0)=u_x^0, & x\in\mathbb{Z}. \end{cases} \tag{2.1}$$

In this section, we summarize some basic properties of solutions to (2.1). The statements presented here are generalizations of the results from our paper [10], which was concerned with the case $m=1$; the proofs for a general $m\in\mathbb{N}$ are straightforward modifications of the original proofs, and we omit them.

In general, the forward solutions of (2.1) need not be unique. However, if the initial condition is bounded, there exists a unique solution of (2.1) which is bounded on every finite time interval. The proof of this fact is similar to the proof of [10, Theorem 3.5].

Theorem 2.1. *If $u^0\in\ell^\infty(\mathbb{Z})$, there exists a unique solution $u:\mathbb{Z}\times[0,\infty)_{\mathbb{T}}\to\mathbb{R}$ of (2.1) which is bounded on every interval $[0,T]_{\mathbb{T}}$, where $T\in[0,\infty)_{\mathbb{T}}$.*

According to the next theorem, bounded solutions of (2.1) preserve space sums whenever the coefficients a_i add up to zero. The proof is the same as the proof of [10, Theorem 4.1] with obvious modifications.

Theorem 2.2. *Assume that*

$$\sum_{i=-m}^{m} a_i = 0. \tag{2.2}$$

If $u:\mathbb{Z}\times[0,T]_{\mathbb{T}}\to\mathbb{R}$ is the unique bounded solution of (2.1) and the sum $\sum_{x\in\mathbb{Z}}|u(x,0)|$ is finite, then

$$\sum_{x\in\mathbb{Z}} u(x,t)=\sum_{x\in\mathbb{Z}} u(x,0),\quad t\in[0,T]_{\mathbb{T}}.$$

For time scales with a sufficiently fine graininess, the solutions of (2.1) preserve the sign of the initial condition; this is a straightforward generalization of [10, Lemma 4.3 and Corollary 4.4].

Theorem 2.3. *Assume that*

$$a_0\le 0,\qquad a_i\ge 0\quad \text{for } i\ne 0, \tag{2.3}$$

$$\mu(t)\le\frac{1}{|a_0|},\quad t\in[0,T)_{\mathbb{T}}. \tag{2.4}$$

If $u_x^0\ge 0$ for every $x\in\mathbb{Z}$ and $u:\mathbb{Z}\times[0,T]_{\mathbb{T}}\to\mathbb{R}$ is the unique bounded solution of (2.1), then $u(x,t)\ge 0$ for all $t\in[0,T]_{\mathbb{T}}$, $x\in\mathbb{Z}$.

Finally, we have the following maximum and minimum principles; see the proof of [10, Theorem 4.7].

Theorem 2.4. *Assume that (2.2)–(2.4) hold. If $u:\mathbb{Z}\times[0,T]_{\mathbb{T}}\to\mathbb{R}$ is the unique bounded solution of (2.1), then*

$$\inf_{y\in\mathbb{Z}} u(y,0)\le u(x,t)\le\sup_{y\in\mathbb{Z}} u(y,0),\quad x\in\mathbb{Z},\ t\in[0,T]_{\mathbb{T}}.$$

3. Continuous dependence

In this section, we consider sequences of time scales $\{\mathbb{T}_n\}_{n=1}^\infty$ such that $\mathbb{T}_n\to\mathbb{T}_0$ in the sense described below. If $u_n:\mathbb{Z}\times\mathbb{T}_n\to\mathbb{R}$ are the corresponding solutions of (1.1), we prove that $u_n\to u_0$ (for ordinary dynamic equations, the