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Existence of solutions for second-order periodic-integrable boundary value problems

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ABSTRACT

The existence of solutions for a second-order periodic-integrable boundary value problem is discussed by a variational method, and the application to the forced oscillation equation is given.

1. Introduction

In this paper, we discuss the second-order equation

$$u'' + f(t, u) = 0, \tag{1.1}$$

with the periodic-integrable boundary value condition

$$u(0) = u(2\pi), \qquad \int_0^{2\pi} u(s)ds = 0, \tag{1.2}$$

where $t \in [0, 2\pi]$, $u \in \mathbb{R}$, $f \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$.

Recently, integral boundary value problems have been studied by many authors; see [1–6]. In these works, some different methods such as lower and upper solution method, monotone iterative method and fixed point method are applied to the problems under consideration. In particular, Hong et al. [1] studied the uniqueness of solutions for the periodic-integrable boundary value problem (PIBVP for short) (1.1) and (1.2) by bilinear form and Schauder's fixed point theorem. We know that in the resonance case, that is, $f_u(t, u) = N^2$ $(N \in \mathbb{Z}^+)$, PIBVP (1.1) and (1.2) may have infinitely many solutions. For example, if $f(t, u) = N^2u$ for $(t, u) \in [0, 2\pi] \times \mathbb{R}$, then PIBVP (1.1) and (1.2) has solutions of the form $u(t) = c_1 \cos(Nt) + c_2 \sin(Nt)$, where c_1, c_2 are two arbitrary constants.

It is obvious that in [1] the conditions imposed on $f(t, u)$ are restrict. If $f(t, u)$ does not satisfy the conditions of [1], then one cannot assure that PIBVP (1.1) and (1.2) has solutions. On the other hand, [7] introduced the basic theory of variational method and discussed some specific examples. Motivated by the above-mentioned questions and the works of [7], in this paper, we discuss the existence of solutions for PIBVP (1.1) and (1.2) by a variational method. Our results improved the one

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by [7] and supplemented the result of [1]. To be clear, we state the main theorem of [1] and the result of [7] cited by this paper as follows:

Theorem A *([1]). Assume that* (A1) *and* (A2) *are satisfied. Then PIBVP* (1.1) *and* (1.2) *has a unique solution.*

(A1) $f \in C^1([0, 2\pi] \times \mathbb{R}, \mathbb{R})$;
(A2) *there exist* $N \in Z^+$ *and* $\epsilon > 0$ *such that* $N^2 + \varepsilon \le f_u(t, u) \le (N+1)^2 - \varepsilon$ *for all* $(t, u) \in ([0, 2\pi] \times \mathbb{R})$.

Theorem B *([7]). Assume that* $e(t) \in C[0, T]$ *is a T-periodic function with* $\int_0^T e(t)dt = 0$ *and a is a constant. Then the forced oscillation equation*

$$u''(t) + a \sin u(t) = e(t)$$

has at least one T-periodic solution.

2. Existence of solutions

We shall use the following conditions:

(C1) $f(t, u) \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$ is 2π-periodic in t for $u \in \mathbb{R}$;
(C2) $\lim_{u\to\infty} \frac{F(t,u)}{u^2} = l < \frac{1}{2}$ for $t \in [0, 2\pi]$.

It is easy to see that a 2π-periodic solution of (1.1) with zero mean value must be a solution of PIBVP (1.1) and (1.2), by which, we shall give the existence result of PIBVP (1.1) and (1.2) by finding the minimum point of a functional.

Let $H = H^1_{per}((0, 2\pi), \mathbb{R})$, that is the closure of 2π-periodic functions belonging to C^∞ in the space $H^1(0, 2\pi) = \{u \in H^1((0, 2\pi), \mathbb{R}) | u(0) = u(2\pi)\}$. Then, a solution $u \in H$ of Eq. (1.1) under the condition $\int_0^{2\pi} u(s)ds = 0$ is a solution of PIBVP (1.1) and (1.2). Define the functionals by

$$I(u) = \int_0^{2\pi} \left(\frac{1}{2} |u'(t)|^2 - F(t, u(t)) \right) dt, \tag{2.1}$$

and

$$N(u) = \int_0^{2\pi} u(s)ds,$$

where

$$F(t, u) = \int_0^u f(t, s)ds. \tag{2.2}$$

Then under the constraint condition $N(u) = 0$, the minimum point $u^* \in C^2(0, 2\pi) \cap H$ of the functional $I(u)$ is the solution of PIBVP (1.1) and (1.2), that is, $I(u^*) = \min_{u\in H\cap N^{-1}(0)} I(u)$. It is easy to see that the Lagrange multiplier vanishes under the condition $N(u) = 0$ (see [7]), so we still consider the minimum point of the functional $I(u)$. However, the functional $I(u)$ is not coercive on H, by the similar idea to [7] we pay our attention to the minimum of the functional $I(u)$ on the weakly closed subspace $H_s := \{u : u \in H, \bar{u} = \int_0^{2\pi} u(s)ds = 0\}$ of H. To begin with, we list some Lemmas as follows.

Lemma 2.1 *(Wirtinger Inequality [7]). If* $u \in H^1_{per}(0, T)$ *and* $\bar{u} = \frac{1}{T} \int_0^T u(t)dt = 0$, *then*

$$\int_0^T |u|^2 dt \le \frac{T^2}{4\pi^2} \int_0^T |u'|^2 dt.$$

With the help of the Lemma 2.1, on the subspace H_s, *we can obtain the equivalent norm*

$$\|u\| = \left(\int_0^{2\pi} |u'(t)|^2 dt \right)^{\frac{1}{2}}.$$

Lemma 2.2 *([7]). Let* $u^* \in W^{1,r}(J, \mathbb{R}^N)$ *be a minimum point of the functional* $I(u) = \int_J L(t, u, u')dt$, *where* $J = [a, b]$ *is a finite interval on* $\mathbb{R}$, $1 < r < \infty$. *Suppose the following two conditions are satisfied*

(i)

$$|L(t, u, p)| + |L_u(t, u, p)| + |L_p(t, u, p)| \le C(1 + |p|^r),$$

(ii) *The matrix* $(L_{p_i p_j}(t, u, p))$, $\forall (t, u, p) \in \bar{J} \times \mathbb{R}^N \times \mathbb{R}^N$ *is positive.*

Then $u^* \in C^2$ in the sense of changing the value of u^* in a set with zero measure.