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Two new regularity criteria for the Navier–Stokes equations via two entries of the velocity Hessian tensor

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A B S T R A C T

We consider the Cauchy problem for the incompressible Navier–Stokes equations in $\mathbb{R}^3$, and provide two sufficient conditions to ensure the smoothness of solutions. Both of them only involve two entries of the velocity Hessian tensor.

1. Introduction

This paper is concerned with the global regularity of solutions of the three-dimensional Navier–Stokes equation (NSE):

$$\begin{cases} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p = \boldsymbol{0}, & \text{in } \mathbb{R}^3 \times (0,T), \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \mathbb{R}^3\times(0,T), \\ \boldsymbol{u} = \boldsymbol{u}_0, & \text{on } \mathbb{R}^3 \times \{t=0\}, \end{cases} \tag{1.1}$$

where $T > 0$ is a given time, $\boldsymbol{u} = (u_1, u_2, u_3)$ is the velocity field, p is a scalar pressure, and $\boldsymbol{u}_0$ is the initial velocity field satisfying $\nabla\cdot\boldsymbol{u}_0 = 0$ in the sense of distributions.

The global existence of a weak solution $\boldsymbol{u}$ to (1.1) with initial data of finite energy is well-known since the work of Leray [1], see also Hopf [2]. However, the issue of uniqueness and regularity of $\boldsymbol{u}$ was left open, and is still unsolved up to date. Pioneered by Serrin [3] and Prodi [4], there have been a lot of literatures devoted to finding sufficient conditions to ensure the smoothness of solutions. Instead of the whole components, here we are interested in regularity criteria in terms of one/partial components. For recent studies in this direction, we refer to [5–18] and references therein.

Motivated by [11,13], in this paper, we consider the regularity criteria involving $\partial_1\partial_3 u_3$, $\partial_2\partial_3 u_3$, or $\partial_1\partial_2 u_1$, $\partial_1\partial_2 u_2$ only.

Now our regularity criteria read:

Theorem 1.1. *Let $\boldsymbol{u}_0 \in H^1(\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{u}_0 = 0$. For $T > 0$, suppose the corresponding strong solution $\boldsymbol{u}$ of (1.1) satisfies*

$$\partial_1\partial_3 u_3,\ \partial_2\partial_3 u_3 \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \qquad \frac{2}{\alpha} + \frac{3}{\beta} = 2 + \frac{1}{\beta}, \quad 1 < \beta \le \infty, \tag{1.2}$$

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or

$$\partial_1\partial_2 u_1,\, \partial_1\partial_2 u_2 \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \qquad \frac{2}{\alpha}+\frac{3}{\beta}=2+\frac{1}{\beta}, \quad 1<\beta\le\infty, \tag{1.3}$$

then $\boldsymbol{u}$ is smooth on $(0,T]$.

Theorem 1.1 under condition (1.2) will be proved in Section 2, and the case (1.3) will be treated in Section 3. Before doing that, we collect here some notations used throughout this paper and make some remarks on our result.

The usual Lebesgue spaces $L^q(\mathbb{R}^3)$ $(1\le q\le\infty)$ are endowed with the norm $\|\cdot\|_q$. For a Banach space $(X,\|\cdot\|)$, we do not distinguish it with its vector analogues X^3, thus the norm in X^3 is still denoted by $\|\cdot\|$; however, all vector- and tensor-valued functions are printed boldfaced. We also denote by

$$\partial_i\varphi=\frac{\partial\varphi}{\partial x_i}, \qquad \partial_{ij}^2\varphi=\partial_i\partial_j\varphi=\frac{\partial^2\varphi}{\partial x_i\partial x_j}, \quad 1\le i\le 3$$

the first- and second-order derivatives of a function φ; by

$$\nabla_h\varphi=(\partial_1,\partial_2)\,\varphi, \qquad \Delta_h\varphi=\left(\partial_{11}^2+\partial_{22}^2\right)\varphi$$

the horizontal gradient, and horizontal Laplacian of φ.

2. Proof of Theorem 1.1 with (1.2)

In this section, we shall prove Theorem 1.1 in case (1.2). First, let us recall and prove some technical lemmas.

The first one being a component-reducing technique due to Kukavica and Ziane [9].

Lemma 2.1. *Assume $\boldsymbol{u}=(u_1,u_2,u_3)\in C_c^\infty(\mathbb{R}^3)$ is divergence free. Then*

$$\sum_{i,j=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i u_j\Delta_h u_j\,\mathrm{d}x=\frac{1}{2}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\partial_i u_j\partial_i u_j\partial_3 u_3\,\mathrm{d}x-\int_{\mathbb{R}^3}\partial_1 u_1\partial_2 u_2\partial_3 u_3\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_1 u_2\partial_2 u_1\partial_3 u_3\,\mathrm{d}x. \tag{2.1}$$

And next, let us prove a variant of an inequality in [11].

Lemma 2.2. *Let $1<r\le 3$, $\{i,j,k\}$ be a permutation of $\{1,2,3\}$. Assume $f,\ g,\ h\in C_c^\infty(\mathbb{R}^3)$. Then there exists a constant $C>0$ such that*

$$\int_{\mathbb{R}^3} fgh\mathrm{d}x\le C\,\|f\|_2^{\frac{r-1}{r}}\,\|\partial_i f\|_{\frac{2}{3-r}}^{\frac{1}{r}}\,\|g\|_2^{\frac{r-1}{r}}\,\left\|\partial_j g\right\|_2^{\frac{1}{2r}}\,\|\partial_k g\|_2^{\frac{1}{2r}}\,\|h\|_2^{\frac{r-1}{r}}\,\left\|\partial_j h\right\|_2^{\frac{1}{2r}}\,\|\partial_k h\|_2^{\frac{1}{2r}}\,.$$

Proof. Without loss of generality, we assume $i=3, j=1$ and $k=2$.

$$\begin{aligned}
\int_{\mathbb{R}^3} fgh\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 &\le \int_{\mathbb{R}^2}\max_{x_3}|f|\cdot\left(\int_{\mathbb{R}}|g|^2\,\mathrm{d}x_3\right)^{1/2}\cdot\left(\int_{\mathbb{R}}|h|^2\,\mathrm{d}x_3\right)^{1/2}\mathrm{d}x_1\mathrm{d}x_2\\
&\le\left[\int_{\mathbb{R}^2}\left(\max_{x_3}|f|\right)^r\mathrm{d}x_1\mathrm{d}x_2\right]^{1/r}\cdot\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|g|^2\,\mathrm{d}x_3\right)^{\frac{r}{r-1}}\mathrm{d}x_1\mathrm{d}x_2\right]^{\frac{r-1}{2r}}\\
&\quad\cdot\left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}|h|^2\,\mathrm{d}x_3\right)^{\frac{r}{r-1}}\mathrm{d}x_1\mathrm{d}x_2\right]^{\frac{r-1}{2r}}\\
&\le C\left[\int_{\mathbb{R}^3}|f|^{r-1}\,|\partial_3 f|\,\mathrm{d}x\right]^{1/r}\cdot\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|g|^{\frac{2r}{r-1}}\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{r-1}{r}}\mathrm{d}x_3\right]^{1/2}\\
&\quad\cdot\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|h|^{\frac{2r}{r-1}}\,\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{r-1}{r}}\mathrm{d}x_3\right]^{1/2}\\
&\le C\,\|f\|_2^{\frac{r-1}{r}}\,\|\partial_3 f\|_{\frac{2}{3-r}}^{\frac{1}{r}}\,\|g\|_2^{\frac{r-1}{r}}\,\|\partial_1 g\|_2^{\frac{1}{2r}}\,\|\partial_2 g\|_2^{\frac{1}{2r}}\,\|h\|_2^{\frac{r-1}{r}}\,\|\partial_1 h\|_2^{\frac{1}{2r}}\,\|\partial_2 h\|_2^{\frac{1}{2r}}\,. \quad\square
\end{aligned}$$

Now, we go to the proof of Theorem 1.2.

Step 1. Preliminary reduction.

It is well-known that the strong solution exists locally, say $\boldsymbol{u}\in C(0,\Gamma^*), H^1(\mathbb{R}^3)\cap L^2(0,\Gamma^*;H^2(\mathbb{R}^3))$. If $\Gamma^*>T$, we have already that $\boldsymbol{u}$ is regular on $(0,T]$. In case $\Gamma^*\le T$, our strategy is to show that $\|\nabla\boldsymbol{u}(t)\|_2$ remains bounded independently of