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A modified Bakhvalov mesh

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ABSTRACT

We consider a few numerical methods for solving a one-dimensional convection–diffusion singularly perturbed problem. More precisely, we introduce a modified Bakvalov mesh generated using some implicitly defined functions. Properties of this mesh and convergence results for several methods on it are given. Numerical results are presented in support of the theoretical considerations.

1. Introduction

We consider the singularly perturbed boundary value problem

$$-\varepsilon u'' - b(x)u' + c(x)u = f(x) \quad \text{in } (0,1), \qquad u(0) = u(1) = 0, \tag{1}$$

assuming that $0 < \varepsilon \ll 1$, $b(x) > 1$ for $x \in [0,1]$ and that functions b, c, f are sufficiently smooth. Then, u can be decomposed as follows: $u = S + E$, where the smooth part S and the layer component E satisfy

$$|S^{(k)}| \le C, \qquad |E^{(k)}| \le C\varepsilon^{-k} e^{-x/\varepsilon} \quad k = 0, 1, \ldots, m \text{ (for any prescribed } m),$$

where C is a generic constant independent of ε.

For solving (1) numerically, Bakhvalov [1] defined a fine mesh in the layer region at $x = 0$ by

$$q(1 - e^{-x_i/(\sigma\varepsilon)}) = \frac{i}{N}.$$

Away from the layer, an equidistant mesh is used with a transition point τ such that the resulting mesh generating function is from C^1, i.e.,

$$\varphi(t) = \begin{cases} -\sigma\varepsilon \ln \dfrac{q-t}{q} := \chi(t) & \text{for } t \in [0, \tau] \\ \text{linear} & \text{for } t \in [\tau, 1]. \end{cases}$$

τ solves a nonlinear equation. Bakhvalov type meshes use some approximations $\tilde{\tau}$ with $\chi(\tilde{\tau}) = \sigma\varepsilon \ln(\frac{q}{\sigma\varepsilon})$; see [2].

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It is practically inconvenient to use different mesh generating functions in different regions. Therefore, we propose to generate a mesh using the implicitly defined function

$$\xi(t) - e^{-\xi(t)/(\sigma\varepsilon)} + 1 - 2t = 0 \tag{2}$$

and to set

$$x_i = \xi(i/N), \quad i = 0, 1, \ldots, N-1, \qquad x_N = 1. \tag{3}$$

We shall study the properties of the mesh based mostly on (2). Practically, one can use the Lambert W-function to generate the mesh, because (see [3]) we have the explicit representation

$$x_i = \frac{2i}{N} - 1 + \varepsilon\sigma W\left(\frac{1}{\varepsilon\sigma} e^{\frac{1-\frac{2i}{N}}{\varepsilon\sigma}}\right).$$

Remark that $x = -b - W(-a^{-b} \ln a)/\ln a$ solves $a^x = x + b$. The W-function is, e.g., in MATLAB and *Mathematica* available.

The parameter σ determines the grading of the mesh inside the layer. In general, we use $\sigma = l$ (or $l + 1$) in combination with a discretization method of order l.

This mesh was already described in [4], but without obtaining any uniform convergence result. In contrast to that paper we present as well optimal convergence estimates for some difference methods as almost optimal error estimates for finite elements. Moreover, the authors of [4] compared the new mesh with the much simpler Shishkin mesh. This is not fair: it makes only sense to compare it with the Bakhvalov mesh.

2. Properties of the modified Bakhvalov mesh

Let us assume N to be even. The point $x_{N/2}$ solves

$$x_{N/2} = e^{-x_{N/2}/(\sigma\varepsilon)} \quad \text{or} \quad x_{N/2} = \varepsilon\sigma W\left(\frac{1}{\sigma\varepsilon}\right).$$

Therefore one gets (if we use the second representation we need $W(z) = \ln z + o(\ln z)$ for $z \to \infty$)

$$x_{N/2} = \varepsilon\sigma\left[\ln\left(\frac{1}{\varepsilon\sigma}\right) + o\left(\ln\left(\frac{1}{\varepsilon\sigma}\right)\right)\right].$$

Differentiation of (2) yields

$$\xi' = \frac{2}{1 + \frac{1}{\varepsilon\sigma}e^{-\xi/(\varepsilon\sigma)}} \quad \text{and} \quad \xi'' = \frac{4}{(\varepsilon\sigma)^2} \cdot \frac{e^{-\xi/(\varepsilon\sigma)}}{\left(1 + \frac{1}{\varepsilon\sigma}e^{-\xi/(\varepsilon\sigma)}\right)^3}.$$

Consequently, $h_i = x_{i+1} - x_i$ increases monotonically. Because $\xi'(1/2) > 1$, the step sizes satisfy

$$2N^{-1} > h_i \geq N^{-1} \quad \text{for } i \geq \frac{N}{2}.$$

x_1 satisfies

$$x_1 + 1 - \frac{2}{N} = e^{-x_1/(\varepsilon\sigma)}.$$

It follows

$$x_1 = C\varepsilon(N^{-1} + o(N^{-1}))$$

as usual for the first mesh point of a layer-adapted mesh.

The definitions (2), (3) imply

$$e^{-x_i/(\varepsilon\sigma)} - e^{-x_{i+1}/(\varepsilon\sigma)} = \frac{2}{N} - (x_{i+1} - x_i) < \frac{2}{N}.$$

3. Convergence results for the modified Bakhvalov mesh

Linß [2] introduced the quantity

$$\vartheta^{[p]} := \max_{i=1,\ldots,N} \int_{x_{i-1}}^{x_i} (1 + \varepsilon^{-1}e^{-s/(p\varepsilon)})ds$$

and proved in Chapter 4 of [2] uniform convergence results for several difference schemes, including convergence acceleration techniques. For simple upwinding one has in the discrete maximum norm

$$\|u - u^N\|_{\infty,d} \leq C\vartheta^{[1]},$$