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Nonexistence of positive solutions for a class of p-Laplacian boundary value problems

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ABSTRACT

We prove the nonexistence of positive radial solutions for the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_p denotes the p-Laplacian, $p > 1$, Ω is a ball or an annulus in $\mathbb{R}^N$, $N > 1$, $f : [0, \infty) \to \mathbb{R}$ is at least p-linear, $f(0) < 0$, and is not required to be increasing or to have exactly one zero. Our results extend previous nonexistence results in the literature.

1. Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a ball or an annulus in $\mathbb{R}^N$, $N > 1$, $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz continuous and $f(0) < 0$.

Problems of the type (1.1) occur in some physical models such as non-Newtonian fluids and chemical reactions, see e.g. [1,2]. In non-Newtonian fluids, the case $p \in (1, 2)$ represents pseudo-plastic while $p > 2$ corresponds to dilatant fluids. The case $p = 2$ represents Newtonian fluids.

We are interested in studying nonexistence of positive solutions to (1.1) in the non-positone case i.e. $f(0) < 0$. When $\Omega = B(0, R)$, a ball with radius R, nonnegative solutions to (1.1) with $f(0) \neq 0$ are positive, radially symmetric and decreasing [3], thus solve the ODE problem

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1} f(u), & 0 < r < R, \\ u'(0) = 0, \quad u(R) = 0, \end{cases} \tag{1.2}$$

where $\phi(u) = |u|^{p-2}u$.

When $f(0) < 0$ and f is p-superlinear at ∞ i.e. $\lim_{u\to\infty} f(u)/u^{p-1} = \infty$, the existence of a positive solution to (1.2) for λ small was established in [4,5], while nonexistence result for λ large was considered in [6] under the assumptions that f is nondecreasing, has exactly one zero, and $\lim_{s\to\infty} f(s)/s^q > 0$ for some $q > p - 1$. Related nonexistence results for positive

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radial solutions to (1.1) in the annulus $\Omega = B(0, R) \setminus \overline{B(0, R_0)}$ i.e.

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1} f(u), & 0 < R_0 < r < R, \\ u(R_0) = 0, \quad u(R) = 0, \end{cases} \tag{1.3}$$

can be found in [7–9] in the case $p = 2$.

In this note, we shall improve the nonexistence results in [7–9,6] by allowing f to be non-monotone with more than one zero and $\liminf_{s\to\infty} f(s)/s^{p-1} > 0$. Our results also complement an existence result to (1.3) for λ small and $p > 1$ in [10]. To be precise, we shall make the following assumptions:

(A1) $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz continuous and $f(0) < 0$.

(A2) There exists a constant $A > 0$ such that $F(s) < 0$ for $0 < s \le A$ and $f(s) > 0$ for $s \ge A$, where $F(s) = \int_0^s f(t)dt$.

(A3) $\liminf_{s\to\infty} f(s)/s^{p-1} > 0$.

Our main results are

Theorem 1.1. *Let $\Omega = B(0, R)$ and suppose* (A1)–(A3) *hold. Then there exists a constant $\lambda_0 > 0$ such that problem* (1.1) *has no nonnegative solutions for $\lambda > \lambda_0$.*

Theorem 1.2. *Let $\Omega = B(0, R) \setminus \overline{B(0, R_0)}$, $0 < R_0 < R$. Suppose* (A1)–(A3) *hold and $p \in (1, 2]$. Then there exists a constant $\lambda_0 > 0$ such that problem* (1.1) *has no positive radial solutions for $\lambda > \lambda_0$.*

Example 1.1. Let $f(s) = (s^\gamma - a)(s^\gamma - b)(s^\gamma - c)$, where γ, a, b, c are positive constants with $\gamma \ge (p-1)/3$, $a < b < c$ and $b < a(\gamma + 1)$. Clearly, (A1) and (A3) hold. Since $f(s) < (b - a)(c - a)(s^\gamma - a)$ for $s \in (0, b^{1/\gamma}]$, $s \ne a^{1/\gamma}$, it follows that

$$F(b^{1/\gamma}) < (b - a)(c - a) \int_0^{b^{1/\gamma}} (s^\gamma - a)ds = (b - a)(c - a)b^{1/\gamma} \left(\frac{b}{\gamma + 1} - a \right) < 0,$$

which implies $F(s) < 0$ for $s \in (0, c^{1/\gamma}]$ and so (A2) holds for some $A > c^{1/\gamma}$. Hence Theorem 1.1 holds for $p > 1$ while Theorem 1.2 holds for $p \in (1, 2]$. Note that f is non-monotone with exactly three zeros on $(0, \infty)$, and $\lim_{s\to\infty} f(s)/s^{p-1} \in (0, \infty)$ for $\gamma = (p - 1)/3$.

2. Proof of the main results

In view of (A1), (A3) and the fact that $\lim_{s\to 0^+} F(s)/s = f(0) < 0$, there exist constants $k, K > 0$ such that

$$-F(s) \ge ks \quad \text{for } s \in (0, A] \tag{2.1}$$

and

$$f(s) \ge K \quad \text{for } s \ge A. \tag{2.2}$$

Note that if u is a solution of (1.2) or (1.3) then

$$\left(\frac{p - 1}{p} |u'|^p + \lambda F(u) \right)' = -\frac{N - 1}{r} |u'|^p \le 0,$$

which implies

$$|u'|^p \ge -\lambda c_p F(u) \tag{2.3}$$

for all r, where $c_p = p/(p - 1)$.

Proof of Theorem 1.1. Let u be a nonnegative solution to (1.1). Then u is radially symmetric, positive, and decreasing [3, Theorem 1] and thus solves (1.2).

We claim that $u(R/4) > A$ for $\lambda \gg 1$. Suppose $u(R/4) \le A$. Then $u \le A$ on $[R/4, R)$, which together with (2.1) and (2.3), implies

$$-u' \ge (\lambda k c_p u)^{1/p},$$

or, equivalently,

$$-\frac{u'}{u^{1/p}} \ge (\lambda k c_p)^{1/p} \quad \text{on } [R/4, R). \tag{2.4}$$

Integrating (2.4) on $[R/4, 3R/4]$ gives

$$u^{\frac{p-1}{p}}(R/4) \ge (R/2)(\lambda k c_p)^{1/p} c_p^{-1} > A$$

for λ large, a contradiction which proves the claim. Hence, by (2.2),

$$-(r^{N-1}\phi(u'))' = \lambda r^{N-1} f(u) \ge \lambda r^{N-1} K \quad \text{on } [0, R/4].$$