Applied Mathematics Letters 31 (2014) 18–24

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Transforms from differential equations to difference equations and vice-versa applied to computer control systems☆

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ARTICLE INFO

Article history:
Received 30 September 2013
Received in revised form 17 December 2013
Accepted 17 December 2013
Available online 20 January 2014

Keywords:
Differential equation
Difference equation
Continuous-time system
Discrete-time system
Bilinear transform
z-*s* transform

ABSTRACT

This letter derives the transform relationship between differential equations to difference equations and vice-versa, applied to computer control systems. The key is to obtain the rational fraction transfer function model of a time-invariant linear differential equation system, using the Laplace transform, and to obtain the impulse transfer function model of a time-invariant linear difference equation, using the shift operator. Finally, we find the discrete-time models of the first-order, second-order and third-order systems from their continuous-time models and vice-versa and find the mapping relationship between the coefficients of discrete-time models and the continuous-time models using the bilinear transform. An example is provided to demonstrate the proposed model transform methods.

1. Introduction

Consider the continuous-time system described by a time-invariant linear differential equation [1]:

$$y^{(n)}(t)+a_1y^{(n-1)}(t)+\cdots+a_{n-1}y^{(1)}(t)+a_ny(t)=b_1u^{(n-1)}(t)+b_1u^{(n-1)}(t)+\cdots+b_{n-1}u^{(1)}(t)+b_nu(t), \tag{1}$$

where $u(t)$ and $y(t)$ denote the input and output of the system, respectively, t represents the time variable, and a_i and b_i are the parameters of this system.

Define the Laplace transform of the integrable function $f(t)\in\mathbb{R}$ as follows:

$$F(s):=\mathscr{L}[f(t)]=\int_0^{+\infty}f(t)\mathrm{e}^{-st}\mathrm{d}t.$$

Under the zero initial values, taking the Laplace transform to both sides of (1) gives

$$(s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n)Y(s)=(b_0s^n+b_1s^{n-1}+b_2s^{n-2}+\cdots+b_{n-1}s+b_n)U(s).$$

Thus, we can obtain the transfer function of the system:

$$G(s):=\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}. \tag{2}$$

☆ This work was supported by the National Natural Science Foundation of China (No. 61273194) and the PAPD of Jiangsu Higher Education Institutions and the 111 Project (B12018).

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http://dx.doi.org/10.1016/j.aml.2013.12.011

Consider the discrete-time system described by a time-invariant linear difference equation:

$$y(k)+\alpha_1y(k-1)+\alpha_2y(k-2)+\cdots+\alpha_ny(k-n)=\beta_0u(k)+\beta_1u(k-1)+\beta_2u(k-2)+\cdots+\beta_nu(k-n), \quad (3)$$

where $u(k):=u(t)|_{t=kh}$ and $y(k):=y(t)|_{t=kh}$, and h represents the sampling period.

Let z denote a unit forward shift operator or the Z transform operator with $zx(k)=x(k+1)$ and $z^{-1}x(k)=x(k-1)$. Eq. (3) can be equivalently written as

$$(1+\alpha_1z^{-1}+\alpha_2z^{-2}+\cdots+\alpha_nz^{-n})Y(z)=(\beta_0+\beta_1z^{-1}+\beta_2z^{-2}+\cdots+\beta_nz^{-n})U(z).$$

Its impulse transfer function is given by

$$\begin{aligned}H(z)&:=\frac{Y(z)}{U(z)}=\frac{\beta_0+\beta_1z^{-1}+\beta_2z^{-2}+\cdots+\beta_nz^{-n}}{1+\alpha_1z^{-1}+\alpha_2z^{-2}+\cdots+\alpha_nz^{-n}}\\&=\frac{\beta_0z^n+\beta_1z^{n-1}+\beta_2z^{n-2}+\cdots+\beta_n}{z^n+\alpha_1z^{n-1}+\alpha_2z^{n-2}+\cdots+\alpha_n}.\end{aligned} \quad (4)$$

The integer number n in $G(s)$ and $H(z)$ is called the system order [2].

Practical systems are generally controlled by using digital computers. Continuous-time signals are sampled periodically and continuous-time systems are often transformed into their corresponding discrete-time systems for control requirements [3]. In the area of system identification, discrete-time models are often used for parameter estimation [4–6]. This letter provides the methods for this purpose and gives the relationship between the coefficients of the continuous-time systems and the discrete-time systems.

The bilinear transform (also known as the Tustin method) is used in digital signal processing [7] and discrete-time control theory [8] to transform continuous-time system representations to discrete-time and vice versa. In practice, most systems are continuous-time systems. With the development of computer technology, physical systems are generally controlled by digital computers [9–11]. So it is necessary to study new methods which transform a continuous-time system into a discrete-time system. Such methods include the Euler transform [12], the step invariance transform or the zero-order hold based transform [13], the bilinear transform or the Tustin transform [14] between s-domain and z-domain, and the well-known $z-s$ transform proposed by Ding [13].

The Euler transform is very simple and easy to implement but the disadvantage is that its transform accuracy is poor and it is difficult to meet the accuracy requirements. The step invariance transform, i.e., the zero-order hold based transform [12] is based on the state space models of dynamic systems and can guarantee that the output of the discrete-time system is equal to that of the continuous-time system at sampling points. Therefore, it is widely used in computer control systems. However, for the linear systems described by the transfer function models (i.e., continuous-time systems), the bilinear transform can transform them into the impulse transfer functions (i.e., discrete-time systems). This letter derives the mapping relationship between the coefficients of the continuous-time systems and the discrete-time systems for the first-order, the second-order and the third-order rational fraction transfer function models.

The rest of this letter is organized as follows. Sections 2 to 4 give the relations of the continuous-time models and the discrete-time models of the first-order, the second-order and the third-order systems and vice-versa according to the bilinear transform. Section 5 offers an example to show that the proposed methods are effective.

2. The first-order systems

2.1. The first-order continuous-time system

Consider the following first-order continuous-time system,

$$G(s)=\frac{b}{s+a}, \quad (5)$$

where the variable s is the Laplace operator, and a and b are the parameters of the system.

It is well-known that the z operator and s operator have the following relation:

$$z=\mathrm{e}^{hs},$$

where $h>0$ denotes the sampling period. Using the Taylor series expansion gives

$$z=\mathrm{e}^{hs}=\frac{\mathrm{e}^{h/2s}}{\mathrm{e}^{h/2s}}=\frac{1+h/2s+\frac{1}{2!}(h/2s)^2+\cdots}{1-h/2s+\frac{1}{2!}(h/2s)^2-\cdots}.$$

Taking the first-order approximation yields

$$z^{-1}=\frac{2-hs}{2+hs}, \quad (6)$$