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Oscillation criteria for second-order dynamic equations on time scales

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ABSTRACT

Erbe's and Hassan's contributions regarding oscillation criteria are interesting in the development of oscillation theory of dynamic equations on time scales. The objective of this paper is to amend these results.

1. Introduction

The increasing interest in oscillatory properties of solutions to dynamic equations on time scales is motivated by their applications in the engineering and natural sciences. We refer the reader to [1–19] and the references cited therein. In [10] and [12], the authors studied the oscillatory behavior of second-order dynamic equations

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)f(x(\tau(t))) = 0 \tag{1.1}$$

and

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(t) = 0, \tag{1.2}$$

respectively, where γ is the quotient of odd positive integers and

(A_1) r and p are positive real-valued rd-continuous functions defined on $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$;
(A_2) the delay function $\tau \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfies $\tau(t) \le t$ and $\lim_{t\to\infty} \tau(t) = \infty$;
(A_3) $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ such that $yf(y) > 0, f(y)/y \ge K > 0$ for $y \ne 0$, where K is a constant.

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A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. Points that are right-scattered and left-scattered at the same time are called isolated. Regarding the time scales that consist of only isolated points; see, for example, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = q^{\mathbb{N}}$, and $\mathbb{T} = 2^{\mathbb{N}}$, etc. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t) := f(\sigma(t))$. Some other concepts related to the notion of time scales; see Bohner and Peterson [6,7].

We assume that solutions of (1.1) (or (1.2)) exist for any $t \in [t_0, \infty)_{\mathbb{T}}$. A solution x of (1.1) (or (1.2)) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, we call it nonoscillatory. Eq. (1.1) (or (1.2)) is said to be oscillatory if all its solutions oscillate.

In order to derive oscillation results for (1.1), Erbe et al. [10] utilized the class of functions as follows: $H \in \mathfrak{H}$ if H is defined for $t_0 \le s \le \sigma(t)$, $t, s \in [t_0, \infty)_{\mathbb{T}}$, $H(t, s) \ge 0$, and satisfies $H(\sigma(t), t) = 0$, $H^{\Delta_s}(t, s) \le 0$ for $t > s \ge t_0$, and for each fixed t, $H^{\Delta_s}(t, s)$ is delta integrable with respect to s. For completeness, we present one of the results in [10] as below.

Theorem 1.1 (*See [10, Theorem 1]*). *Assume that (A_1)–(A_3) are satisfied and let*

$$r^{\Delta}(t) \ge 0, \quad \int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty, \quad \text{and} \quad \int_{t_0}^{\infty} p(t)\tau(t)\Delta t = \infty. \tag{1.3}$$

Suppose further that there exist a function η and a positive, differentiable function δ such that for some $H \in \mathfrak{H}$ and for sufficiently large t_1,

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), t_1)} \int_{t_1}^{t} H(\sigma(t), \sigma(s))\delta^{\sigma}(s)\left[\psi(s) - \phi(t, s)\right]\Delta s = \infty, \tag{1.4}$$

where

$$\phi(t, s) := \frac{1}{4}\left(\frac{\delta(s)}{\delta^{\sigma}(s)}\right)^2 \frac{r(s)A^2(t, s)}{C(s)}, \qquad C(t) := \frac{t}{\sigma(t)},$$

$$\psi(s) := \frac{Kp(s)\tau(s)}{\sigma(s)} - (\eta(s)r(s))^{\Delta} + \frac{sr(s)\eta^2(s)}{\sigma(s)},$$

and

$$A(t, s) := \frac{\delta^{\sigma}(s)C_1(s)}{\delta(s)} + \frac{H^{\Delta_s}(\sigma(t), s)}{H(\sigma(t), \sigma(s))}, \qquad C_1(s) := \frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)} + 2\frac{s\eta(s)}{\sigma(s)}.$$

Then (1.1) is oscillatory.

To prove Theorem 1.1, the authors defined a generalized Riccati substitution

$$w(t) := \delta(t)\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)\eta(t)\right], \tag{1.5}$$

and then derived the following formula; see [10, (2.15)]

$$\int_{t_2}^{t} H(\sigma(t), \sigma(s))\delta^{\sigma}(s)\psi(s)\Delta s \le H(\sigma(t), t_2)w(t_2) - \int_{t_2}^{t} H(\sigma(t), \sigma(s))\frac{C(s)\delta^{\sigma}(s)}{r(s)\delta^2(s)}w^2(s)\Delta s + \int_{t_2}^{t} H(\sigma(t), \sigma(s))A(t, s)w(s)\Delta s. \tag{1.6}$$

Then by this inequality and using the method of completing the square, they obtained

$$\int_{t_2}^{t} H(\sigma(t), \sigma(s))\delta^{\sigma}(s)\psi(s)\Delta s \le H(\sigma(t), t_2)w(t_2) + \int_{t_2}^{t} H(\sigma(t), \sigma(s))\frac{r(s)\delta^2(s)A^2(t, s)}{4C(s)\delta^{\sigma}(s)}\Delta s. \tag{1.7}$$

Note that when the time scale $\mathbb{T}$ considered only contains isolated points, e.g., $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, and $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, etc., completing the square cannot be applied in inequality (1.6) to provide (1.7) since $H(\sigma(t), \sigma(\rho(t))) = H(\sigma(t), t) = 0$.

To present oscillation theorems for (1.2), Hassan [12] employed the class of functions as follows: $H \in \mathfrak{R}$ if $H : [t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ and satisfies $H(t, t) = 0$, $t \ge t_0$, $H(t, s) > 0$, $t > s \ge t_0$. In the following, we give one of the results presented in [12] for the convenience of the reader.