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A study on a (2 + 1)-dimensional and a (3 + 1)-dimensional generalized Burgers equation

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ABSTRACT

A (2 + 1)-dimensional and a (3 + 1)-dimensional generalized Burgers equation are investigated. The Cole–Hopf transformation method is used to carry out this work. Multiple kink solutions are formally derived for each equation.

1. Introduction

The study of Burgers equation is very important in solitary waves theory. Burgers equations are encountered in many fields [1–4] such as fluid mechanics, traffic flows, acoustic transmission, gas dynamics, heat conduction, and structure of shock waves. Many systematic methods are used for studying the nonlinear evolution equations that give rise to solitons. The generalized symmetry method, Painlevé analysis, Pfaffian technique, the inverse scattering method, the Bäcklund transformation method, the conservation law method, and the Hirota bilinear method [5–14] are the most commonly used methods. The Cole–Hopf transformation presents ideal method for the determination of multiple kink solutions of the Burgers equation.

In [1], the (2 + 1)-dimensional Burgers equation

$$v_t = v_{xx} + 2v_x\partial_y^{-1}v_x, \tag{1}$$

and the (2 + 1)-dimensional high-order Burgers equation

$$v_t = 4v_{xxx} + 12v_{xx}\partial_y^{-1}v_x + 12v_x\partial_y^{-1}v_{xx} + 12v_x(\partial_y^{-1}v_x)^2, \tag{2}$$

were presented. In [1], the aforementioned equations were examined by using the singular manifold method.

In [3], the authors employed the Painlevé property and derived the (2 + 1)-dimensional generalized Burgers equation. The derived generalized Burgers equation was proved to be Painlevé integrable given in the form

$$\begin{aligned} u_t &= a_1uu_x + a_2u_{xx} + b_1vv_x + \frac{a_2b_1}{a_1}v_{xy}, \\ u_y &= v_x. \end{aligned} \tag{3}$$

The variable separation approach was used in [3] to derive some solutions of this equation.

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Using the same sense used in [3], we extend Eq. (3) to a $(3+1)$-dimensional generalized Burgers equation given in the form

$$\begin{aligned} u_t &= a_1 u u_x + a_2 u_{xx} + b_1 v v_x + \frac{a_2 b_1}{a_1} v_{xy} + c_1 w w_x + \frac{a_2 c_1}{a_1} w_{xz}, \\ u_y &= v_x, \\ u_z &= w_x, \end{aligned} \tag{4}$$

where a_1, a_2, b_1, and c_1 are non-zero constants.

In this work we will employ the Cole–Hopf transformation method to handle these two Eqs. (3) and (4). We aim to obtain multiple kink solutions for each equation.

2. The $(2+1)$-dimensional generalized Burgers equation

We first consider the $(2+1)$-dimensional generalized Burgers equation

$$u_t = a_1 u u_x + a_2 u_{xx} + b_1 u_y \partial_x^{-1} u_y + \frac{a_2 b_1}{a_1} u_{yy}, \tag{5}$$

obtained from (3) by using $v = \partial_x^{-1} u_y$. To get rid of the inverse operator, we use the potential

$$u = p_x, \tag{6}$$

to carry out (5) to the potential form

$$p_{xt} = a_1 p_x p_{xx} + a_2 p_{xxx} + b_1 p_{xy} p_y + \frac{a_2 b_1}{a_1} p_{xyy}. \tag{7}$$

Substituting

$$p(x, y, t) = e^{k_i x + r_i y - c_i t}, \quad i = 1, 2, 3, \tag{8}$$

into the linear terms of (7) gives the dispersion relation by

$$c_i = -a_2 \left(k_i^2 + \frac{b_1}{a_1} r_i^2 \right), \tag{9}$$

and hence the dispersion variable becomes

$$\theta_i = k_i x + r_i y + a_2 \left(k_i^2 + \frac{b_1}{a_1} r_i^2 \right) t, \quad i = 1, 2, 3. \tag{10}$$

Using the Cole–Hopf transformation method, the multi kink solutions of (7) are assumed to be

$$u(x, y, t) = R \left(\ln f(x, y, t) \right)_x, \tag{11}$$

and therefore

$$p(x, y, t) = R \left(\ln f(x, y, t) \right), \tag{12}$$

where the auxiliary function $f(x, y, t)$ for the single kink solution is given by

$$f(x, y, t) = 1 + e^{k_1 x + r_1 y + a_2 \left(k_1^2 + \frac{b_1}{a_1} r_1^2 \right) t}. \tag{13}$$

Substituting (13) into (7) and solving for R we find

$$R = \frac{2a_2}{a_1}. \tag{14}$$

This in turn gives the solution

$$p(x, y, t) = \frac{2a_2}{a_1} \ln \left(1 + e^{k_1 x + r_1 y + a_2 \left(k_1^2 + \frac{b_1}{a_1} r_1^2 \right) t} \right). \tag{15}$$

Using potential (6) gives the single kink solution

$$u(x, y, t) = \frac{2 a_2 k_1 \, e^{k_1 x + r_1 y + a_2 \left(k_1^2 + \frac{b_1}{a_1} r_1^2 \right) t}}{a_1 \left(1 + e^{k_1 x + r_1 y + a_2 \left(k_1^2 + \frac{b_1}{a_1} r_1^2 \right) t} \right)}. \tag{16}$$

For the two kink solutions we set

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2}. \tag{17}$$