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Regularity criteria for the density-dependent Hall-magnetohydrodynamics

Jishan Fan [a], Tohru Ozawa [b,*]

[a] Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, PR China
[b] Department of Applied Physics, Waseda University, Tokyo, 169-8555, Japan

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ABSTRACT

This paper proves two regularity criteria for the density-dependent Hall-MHD system with positive initial density. We also prove a global nonexistence result for initial density with a high decrease at infinity.

1. Introduction

In this paper, we consider the density-dependent Hall-magnetohydrodynamics system:

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla\left(\pi + \frac{1}{2}|b|^2\right) - \Delta u = b \cdot \nabla b, \tag{1.1}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u + \operatorname{curl}(\operatorname{curl} b \times b) = \Delta b, \tag{1.2}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.3}$$

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.4}$$

$$(\rho, u, b)(\cdot, 0) = (\rho_0, u_0, b_0)(\cdot) \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

$$\lim_{|x|\to\infty} \rho = \tilde{\rho}, \qquad \lim_{|x|\to\infty} (u, b) = (0, 0). \tag{1.6}$$

Here ρ is the density of the fluid, u is the fluid velocity field, π is the pressure, b is the magnetic field, and $\tilde{\rho}$ is a positive constant.

Applications of the Hall-MHD system cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo.

When $\rho = 1$, the density-dependent Hall-MHD system reduces to the Hall-MHD system, which has received many studies [1–8]. The paper [1] gave a derivation of Hall-MHD system from a two-fluids Euler–Maxwell system. Chae–Degond–Liu [4] proved the local existence of smooth solutions. Chae–Lee [2] proved the following regularity criteria

$$u \in L^{\frac{2p}{p-3}}(0, T; L^p) \quad \text{and} \quad \nabla b \in L^{\frac{2s}{s-3}}(0, T; L^s) \quad \text{with } 3 < p,\ s \leq \infty \tag{1.7}$$

* Corresponding author. Tel.: +81 352868487; fax: +81 352868487.
E-mail addresses: fanjishan@njfu.edu.cn (J. Fan), txozawa@waseda.jp (T. Ozawa).

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or

$$u \in L^2(0,T;BMO) \quad \text{and} \quad \nabla b \in L^2(0,T;BMO). \tag{1.8}$$

Here *BMO* is the space of functions of bounded mean oscillations.

When the Hall effect term curl (curl $b \times b$) is neglected, the density-dependent Hall-MHD system reduces to the well known density-dependent MHD system. Abidi–Hmidi [9] and Wu [10] proved the local existence of strong solutions. Fan–Li–Nakamura–Tan [11] show some regularity criteria.

The aim of this paper is to prove some regularity criteria of the system (1.1)–(1.6). We will prove

Theorem 1.1. *Let* $0 < \frac{1}{C} \le \tilde{\rho}, \rho_0 \le C$, $\nabla \rho_0 \in L^m$ *with* $3 < m < \infty$ *and* $u_0, b_0 \in H^2$ *with* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ *in* $\mathbb{R}^3$. *Let* (ρ, u, b) *be a local strong solution to the problem* (1.4)–(1.6). *If* (1.7) *or*

$$u \in L^2(0,T;\dot{B}^0_{\infty,2}) \quad \text{and} \quad \nabla b \in L^{\frac{2s}{s-3}}(0,T;L^s) \quad \text{with } 3 < s \le \infty, \tag{1.9}$$

holds true with $0 < T < \infty$, *then the solution* (ρ, u, b) *can be extended beyond* $T > 0$.

Here $\dot{B}^0_{\infty,2}$ *is the homogeneous Besov space.*

By the same calculations as those in [11], we can prove

Theorem 1.2. *Let the initial data* (ρ_0, u_0, b_0) *satisfy that*

$$\rho \in L^{\frac{6}{5}}, \qquad \rho_0|u_0|^2 \in L^1, \qquad \rho_0 u_0 \in L^1, \qquad \int \rho_0 u_0 dx \neq 0, \qquad b_0 \in L^2.$$

Then, there exists no global-in-time smooth solution to the Cauchy problem (1.4)–(1.6) *with* $\tilde{\rho} = 0$.

We omit the proof, since it is almost identical to that in [11].

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. By very similar calculations as those in [10], we can prove the local existence of strong solutions to the problem (1.4)–(1.6) and thus we omit the details here. We only need to establish a priori estimates. We only need to prove the case (1.7), since the proof of the case (1.9) is similar to (1.7) and [11]. Thus we will assume that (1.7) holds true.

First, thanks to the maximum principle, we see that

$$0 < \frac{1}{C} \le \rho \le C < \infty. \tag{2.1}$$

Testing (1.1) by u and using (1.4) and (1.3), we find that

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2 dx + \int |\nabla u|^2 dx = \int (b\cdot\nabla)b\cdot u dx. \tag{2.2}$$

Testing (1.2) by b and using (1.4), we infer that

$$\frac{1}{2}\frac{d}{dt}\int |b|^2 dx + \int |\nabla b|^2 dx = \int (b\cdot\nabla)u\cdot b dx. \tag{2.3}$$

Summing up (2.2) and (2.3) and using

$$\int (b\cdot\nabla)b\cdot u dx + \int (b\cdot\nabla)u\cdot b dx = 0,$$

we get

$$\frac{1}{2}\frac{d}{dt}\int (\rho|u|^2 + |b|^2)dx + \int (|\nabla u|^2 + |\nabla b|^2)dx = 0.$$

This proves

$$\|(u,b)\|_{L^\infty(0,T;L^2)} + \|(u,b)\|_{L^2(0,T;H^1)} \le C. \tag{2.4}$$