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Variational approach to impulsive evolution equations

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ABSTRACT

This article uses variational method for studying existence and uniqueness of solutions for impulsive evolution equations. The main techniques include Hilbert triple, Sobolev embedding theorem, Galerkin approximation and weak convergence for passing to the limit.

1. Introduction

Impulsive differential equations have been studied for modelling processes with abrupt changes. For impulsive ordinary differential equations see for example [1,2]. In [3] impulsive evolution equations via semigroup and fixed point approach are considered. The variational approach in this problem was initiated in the paper [4] by N.U Ahmed.

In this paper, we adopt the approach of variational method and perturbation. The fundamental idea is to use Galerkin approximation which is used to reduce the infinite dimensional problem (the evolution equation) to a problem in finite dimensional subspace. Then by getting a priori estimates and using theorems of weak convergence and compact embedding, the approximate solutions give, by passing to the limit, a solution of the original problem. This approach was summarised in, for example [5,6].

Consider an interval $I = [0, T]$, $T > 0$ and a finite set of points

$$D = \{t_i \in (0, T), i = 1, 2, \dots, n\}, \quad t_1 < t_2 < \dots < t_n, \qquad t_0 = 0, \qquad t_{n+1} = T.$$

Let $\sigma_i = [t_i, t_{i+1})$, $i = 0, 1, \dots n$. H is a Hilbert space. Consider the impulsive problem:

$$\begin{cases} u' + A(t)u(t) = f(t), \quad t \in I \setminus D, \\ u(0) = u_0 \in H, \\ Ju(t_i) = G(t_i, u(t_i)), \quad t_i \in D. \end{cases} \tag{1}$$

J denotes the jump operator defined as

$$Ju(t_i) = u(t_i) - u(t_i-).$$

Here $u(t_i-)$ denotes left hand side limit of u at t_i.

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Next we give some assumptions concerning the operators and terms in the problem. $a(t; u, v) = (A(t)u, v)$ for all $u, v \in V$:

(A1) $a(t; u, v) \leq \mu\|u\|\|v\|$, where μ is a constant independent of $t \in [0, T], u, v \in V$.

(A2) $t \to a(t; u, v)$ is measurable in $]0, T[$ for all $u, v \in V$.

(A3) $\exists\lambda \in R, \alpha > 0$, $Re\, a(t; u, u) \geq \alpha\|u\|^2, u \in V$.

(A4) $G : I \times H \to H$ is continuous in both variables and bounded on bounded subsets of H.

To borrow the terminology of optimal impulse control, we say that $I \setminus D$ and D are, respectively, continuation set and stopping set. We do not have a free boundary value problem since the moments of impulses in our model are fixed.

2. Preliminaries

H is a separable Hilbert space. V is a dense subspace of H with continuous injection:

$$V \hookrightarrow H$$

as usual, H is identified with its dual H'. We denote evolution triple $\{V, H, V'\}$ with embeddings:

$$V \hookrightarrow H \hookrightarrow V' \tag{2}$$

being continuous and dense. $PWC(I, H)$ denotes the space of bounded piecewise continuous functions $x : I \to H$ such that $x : \sigma_i \to H$ is continuous for every $i = 1, 2, \ldots, n$.

$$\|x\| = \sup\{\|x(t)\|_H, t \in I\}. \tag{3}$$

Define $H^1(I; V, V') = \{u : [0, T] \to V\}$ such that $u \in L^2(I; V)$ and u' exists a.e. on I and $u' \in L^2(I; V')$.

We recall here an important embedding theorem for Sobolev space which will be useful later [6,7]:

Lemma 2.1. *$H^1(I; V, V')$ is a Hilbert space with the norm $\|u\|_1^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2$, and the embedding:*

$$H^1(I; V, V') \hookrightarrow C(I; H) \tag{4}$$

is compact.

In other words, if $u \in L^2(I; V)$ and $u' \in L^2(I; V')$, then u is almost everywhere equal to a function continuous from I into H, that is, $u \in C(I; H)$.

Lemma 2.2 ([8,9] *Aubin–Lions Lemma*)**.** *Let W_0, W, W_1 be Banach spaces with $W_0 \subset W \subset W_1$, assume $W_0 \hookrightarrow W$ is compact and $W \hookrightarrow W_1$ is continuous. Let $1 < p, q < \infty$, W_0 and W_1 be reflexive, and define:*

$$X = \{u \in L^p(I; W_0), u \in L^q(I; W_1)\}. \tag{5}$$

Then the inclusion $X \hookrightarrow L^p(I; W)$ is compact.

3. Main result

Theorem 3.1. *Suppose*

(i) *injection of V into H is compact,*
(ii) *$a(t; u, v)$ satisfies* (A1)–(A3),
(iii) *$f \in L^2(I; V')$,*
(iv) *$u_0 \in H$.*

Then impulsive evolution problem (1) *has a unique solution u such that $u \in L^2(I; V) \cap L^\infty(I; H) \cap PWC(I, H)$.*

Proof. We divide the proof into several steps.

Step 1: Uniqueness

If there exist two different solutions, u_1 and u_2, we argue by simple mathematical induction. Let $w = u_1 - u_2$, then $w_0 = 0$ and $\frac{1}{2}\frac{d}{dt}|w(t)|^2 + a(t; w(t), w(t)) = 0$. Hence on interval σ_0, $\frac{1}{2}|w(t)|^2 + \int_0^t a(s, w(s), w(s))ds = 0$. From coercivity condition we obtain: $\frac{1}{2}|w(t)|^2 \leq 0$, so that $w(t) = 0$ for all $t \in \sigma_0$. This leads to $w(t_1-) = 0$.

Since the jump operator J is single valued we have $w(t_1) = w_0^1 = 0$. By induction we have for all i, $w(t_i) = w_i^0 = 0$, and on each interval σ_i, $t_i \leq t < t_{i+1}$, $\frac{1}{2}|w(t)|^2 + \int_{t_i}^t a(s, w(s), w(s))ds = 0$, $w(t) = 0$ for all $t \in \sigma_i$. Thus the solution, if such exists, must be unique.

Step 2: Galerkin approximation

We project the infinite dimensional system into finite dimensional ODE system:

$$\begin{cases} \left(\dfrac{d}{dt}u_m(t), v_j\right) + a(t; u_m(t), v_j) = (f(t), v_j) \\ u_m(t_i + 0) = (\xi_i)_m. \end{cases} \tag{6}$$

We know that (6) has a unique solution that satisfies $u_m \in C(\sigma_i; V_m)$, $u_m \in L^1(\sigma_i; V_m)$ by the ODE method, see [4].