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Oscillation of fourth order sub-linear differential equations

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ABSTRACT

In this paper, we deal with oscillatory and asymptotic properties of solutions of a fourth order sub-linear differential equation with the oscillatory operator. We establish conditions for the nonexistence of positive and bounded solutions and an oscillation criterion.

1. Introduction

Consider the fourth order nonlinear differential equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0\,, \tag{1}$$

where $0 < \lambda < 1$, the functions q and r are continuous on $\mathbb{R}_+ = [0, \infty)$ and such that $q(t) > 0$ and $r(t) > 0$ for large t. Throughout the paper we assume that the associated second order linear equation

$$h''(t) + q(t)h(t) = 0 \tag{2}$$

is oscillatory. We allow that q can tend to zero or to infinity as $t \to \infty$.

By a solution of (1) we mean a function $x \in C^4(\mathbb{R}_+)$, which satisfies (1) on $\mathbb{R}_+$. A solution is said to be *nonoscillatory* if $x(t) \neq 0$ for large t, otherwise is said to be *oscillatory*. Eq. (1) is *oscillatory* if any of its solution is oscillatory.

A prototype of (1) is the equation

$$x^{(4)}(t) + x''(t) + r(t)|x(t)|^{\lambda}\operatorname{sgn} x(t) = 0 \tag{3}$$

investigated in [1]. The following holds.

Theorem A *([1, Corollary 1.6]). Let $\lambda < 1$. The necessary and sufficient condition for Eq. (3) to be oscillatory is*

$$\int_0^{\infty} t^{\lambda} r(t)\, dt = \infty. \tag{4}$$

Recently, there has been a great deal of interest in studying the oscillatory behavior of Eq. (1); see [2–4] and references contained therein. Under the additional conditions on q, it has been proved that the condition (4) is necessary to be (1) oscillatory, see [4, Theorem 2.1], and that the existence of a bounded asymptotically linear solution implies the existence of an unbounded solution of (1); see [4, Corollary 4.5]. A sufficient condition for oscillation of (1) has been proved in [3] in case $\lambda > 1$.

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Motivated by these results, the aim of this paper is to find conditions ensuring that any eventually positive solution of (1) is unbounded, and to give an oscillation theorem for (1).

2. Preliminaries

We start with the classification of eventually positive solutions of (1).

A function g, defined in a neighborhood of infinity, is said to change a sign, if there exists an increasing sequence $\{t_k\} \to \infty$ such that $g(t_k)g(t_{k+1}) < 0$.

Lemma 1. *Let (2) be oscillatory. Every eventually positive solution x of (1) is one of the following type:*
Type (a) : $x(t) > 0, x'(t) > 0, x''(t) \le 0$ *for large t,*
Type (b) : x'' *changes a sign.*

Proof. From Theorem 2, part (b) and Theorem 2' in [3] it follows that if (2) is oscillatory, then every eventually positive solution x satisfies either $x''(t) \le 0$ or x'' changes a sign. Let $x(t) > 0$ and $x''(t) \le 0$ for large t. If $x'(t) \le 0$, then x is nonincreasing and concave, which is a contradiction with the positivity of x. Hence, $x'(t) > 0$ and x is of Type (a). □

Lemma 2. *Let $\lambda < 1$ and (2) be oscillatory. If x is a solution of (1) such that*

$$0 < x(t) \le \left(\frac{4r(t)}{q^2(t)}\right)^{1/(1-\lambda)} \quad \textit{for large } t, \tag{5}$$

then the function

$$F(t) = -x'''(t)\, x(t) + x'(t)\, x''(t),$$

is nondecreasing for large t, and x is of Type (a).

Proof. Let x be a solution of (1) satisfying (5). Then we have

$$\begin{aligned} F'(t) &= r(t)x^{\lambda+1}(t) + q(t)x''(t)x(t) + \left(x''(t)\right)^2 \\ &= \left[\sqrt{r(t)}x^{\frac{\lambda+1}{2}}(t) + \frac{q(t)}{2\sqrt{r(t)}}x^{\frac{1-\lambda}{2}}(t)x''(t)\right]^2 + \left(x''(t)\right)^2\left[1 - \frac{q^2(t)}{4r(t)}x^{1-\lambda}(t)\right] \ge 0 \end{aligned}$$

for large t, so F is nondecreasing for large t.

Let $x(t) > 0$ for $t \ge T_1 \ge 0$ and suppose x'' changes a sign. Let $\{t_k\}_{k=1}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$, $T_1 \le t_k < \tau_k < t_{k+1}$, $k = 1, 2, \ldots$ be sequences of zeros of x'' tending to ∞ such that

$$x''(t) > 0 \quad \text{on } (t_k, \tau_k),\ k = 1, 2, \ldots. \tag{6}$$

From here and (1) we have $x^{(4)}(t) < 0$ on $[t_k, \tau_k]$, and hence, x''' is decreasing on $[t_k, \tau_k]$. In virtue of (6) and the fact that $x''(t_k) = x''(\tau_k) = 0$, there exist numbers $\xi_k \in (t_k, \tau_k)$ such that $x'''(\xi_k) = 0$, $k = 1, 2, \ldots$. Noting that x''' is decreasing on $[t_k, \tau_k]$, we have

$$x'''(t_k) > 0 \quad \text{and} \quad x'''(\tau_k) < 0\,, \quad k = 1, 2, \ldots.$$

Hence,

$$F(t_k) = -x'''(t_k)\, x(t_k) < 0, \qquad F(\tau_k) = -x'''(\tau_k)\, x(\tau_k) > 0, \quad k = 1, 2, \ldots .$$

This means that F changes its sign at points t_k and τ_k. This is impossible because F is nondecreasing for large t. Thus x'' does not change a sign and by Lemma 1 we get the conclusion. □

3. Main results

We start with the problem of the nonexistence of bounded solutions of (1).

Theorem 1. *Let $\lambda < 1$ and (2) be oscillatory. Assume that*

$$\lim_{t\to\infty} \frac{r(t)}{q(t)} = \infty, \qquad \lim_{t\to\infty} \frac{r(t)}{q^2(t)} = \infty, \tag{7}$$

$$\int_0^{\infty} t^2 r(t)\, dt = \infty. \tag{8}$$

If there exists an eventually positive solution x of (1), then it is unbounded and satisfies

$$\limsup_{t\to\infty} x(t)\left(\frac{q^2(t)}{4r(t)}\right)^{1/(1-\lambda)} \ge 1. \tag{9}$$