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Sampling inequalities for infinitely smooth radial basis functions and its application to error estimates

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ABSTRACT

Recently, Rieger and Zwicknagl (2010) have introduced sampling inequalities for infinitely smooth functions to derive Sobolev-type error estimates. They introduced exponential convergence orders for functions within the native space associated with the given radial basis function (RBF). Our major concern of this paper is to extend the results made in Rieger and Zwicknagl (2010). We derive generalized sampling inequalities for the larger class of infinitely smooth RBFs, including multiquadrics, inverse multiquadrics, shifted surface splines and Gaussians.

1. Introduction

Radial basis function interpolation is a very well-established tool for reconstructing multivariate functions from scattered data. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is *radial* in the sense that $\phi(x) = \Phi(|x|)$, where $|\cdot|$ is the usual Euclidean norm. Recent books present the core of the underlying general mathematics for RBF methods [1,2]. In addition, spectral approximation orders of RBF approximation on the Sobolev space were observed in [3,4]. The setup of RBF interpolation considered in this paper is as follows. Suppose that a continuous function $f \in C(\mathbb{R}^d)$ is known only at a set of scattered points $X := \{x_1, \dots, x_N\}$ in a compact subset Ω in $\mathbb{R}^d$. Let $\Pi_{<m}$ denote the linear space of all polynomials on $\mathbb{R}^d$ of degree at most $m - 1$. Then we construct an interpolant $I_{f,X}$ to f of the form

$$I_{f,X}(x) := \sum_{j=1}^{N} \alpha_j \phi(x - x_j) + \sum_{i=1}^{\ell} \beta_i p_i(x), \tag{1.1}$$

where ϕ is a radial basis function and $p_1, \dots, p_\ell$ is a basis of $\Pi_{<m}$. The coefficients α_j and β_i of $I_{f,X}$ in (1.1) are required to satisfy the interpolation condition $I_{f,X}(x_j) = f(x_j)$ $(j = 1, \dots, N)$. For a given $m > 0$, we require that the set X is $\Pi_{<m}$-unisolvent, that is, if $p \in \Pi_{<m}$ and $p|_X = 0$, then p is identically zero. It guarantees the uniqueness of the interpolant when ϕ is *conditionally positive definite* of order $m \geq 0$ on Ω.

Among many RBFs, this paper considers the case of using infinitely smooth functions, because they can provide spectral approximation orders. Wu and Schaback [5] estimated the L_∞-norm of the error between $I_{f,X}^{(\alpha)}$ and $f^{(\alpha)}$ with $\alpha \in \mathbb{Z}_+^d$ by using

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smooth RBFs for functions f in the associated native space

$$\mathcal{F}_\phi := \left\{ f : |f|_\phi^2 := \int_{\mathbb{R}^d} |\hat{f}(\theta)|^2 \hat{\phi}^{-1}(\theta) d\theta < \infty \right\}.$$

Sampling inequalities are very useful methods for error estimates of interpolation and approximation [6–9]. Recently, in [10], Rieger and Zwicknagl obtained the sampling inequalities for infinitely smooth functions in native spaces of Gaussians and inverse multiquadrics. By using these inequalities, they derived Sobolev-type exponential approximation orders when the function f belongs to $\mathcal{F}_\phi$ and Ω is a compact cube.

The aim of this paper is to extend the result made in [10]. We first derive generalized sampling inequalities and then apply them for larger class of infinitely smooth RBFs, including multiquadrics, inverse multiquadrics, shifted surface splines and Gaussians (see Section 2). Furthermore, Sobolev-type exponential approximation orders of interpolation are investigated for the functions f in the native space $\mathcal{F}_\phi$.

2. Notations and preliminaries

In order to discuss the extent to which RBF interpolation approximates f, we define the *fill-distance* h and the *separation distance* q of X in Ω by

$$h := h_{X,\Omega} := \sup_{x\in\Omega} \min_{1\le j\le N} |x - x_j| \quad \text{and} \quad q := q_X := \min_{1\le i\ne j\le N} |x_i - x_j|/2. \tag{2.1}$$

Throughout this paper, we assume, without great loss, that X is quasi-uniform, i.e., there exists a constant $\eta > 0$ independent of X such that $h/q \le \eta$. This condition asserts that the number of the scattered points in X is bounded by ch^{-d}, i.e., $N \le ch^{-d}$, where the constant $c > 0$ is independent of X.

In this paper, we are particularly interested in using the following infinitely smooth RBFs:

$$\begin{aligned}
&\text{(a) } \phi(x) = (-1)^{\lceil\beta\rceil}(|x|^2+\lambda^2)^\beta, \quad \beta > 0, \beta \notin \mathbb{N}, \text{ (multiquadrics)},\\
&\text{(b) } \phi(x) = (|x|^2+\lambda^2)^\beta, \quad \beta < 0, \text{ (inverse multiquadrics)},\\
&\text{(c) } \phi(x) = (-1)^{\beta+1}(|x|^2+\lambda^2)^\beta \log(|x|^2+\lambda^2)^{1/2}, \quad \beta\in\mathbb{N}, \text{ (shifted surface splines)},\\
&\text{(d) } \phi(x) = e^{-\lambda|x|^2}, \text{ (Gaussians)},
\end{aligned} \tag{2.2}$$

where $\lambda > 0$ and $\lceil s\rceil$ indicates the smallest integer greater than s. Let ϕ be one of the RBFs such as multiquadrics, inverse multiquadrics and shifted surface splines in (2.2). When $\beta \ge -d/2$, the basis function ϕ grows polynomially or decays slowly for large argument such that ϕ is not integrable. Then, in the sense of tempered distribution, ϕ has the generalized Fourier transform

$$\hat{\phi}(\theta) = c_{\beta,\lambda}|\theta|^{-(2\beta+d)_+}\widetilde{K}_{|d/2+\beta|}(\lambda|\theta|), \tag{2.3}$$

for some suitable constant $c_{\beta,\lambda}$ depending on β and λ, where $t_+ := \max(t,0)$, $t\in\mathbb{R}$, and $\widetilde{K}_\nu(t) = t^\nu K_\nu(t)$ with K_ν the modified Bessel function of the third kind of order ν [2,11]. It is well known from the literature (e.g.,[11]) that $\widetilde{K}_\nu(t)\in C(\mathbb{R})$, $\nu > 0$ and $\widetilde{K}_0(t) \sim -\log(t)$ when $t \to 0$. In particular, when $\beta + d/2 < 0$, the inverse multiquadrics have the classical Fourier transform of the form in (2.3).

We will use the notation $\mathbb{Z}_+^d := \{(\gamma_1,\dots,\gamma_d)\in\mathbb{Z}^d : \gamma_k \ge 0\}$ and $|\alpha|_1 := \sum_{k=1}^d \alpha_k$ for $\alpha\in\mathbb{Z}_+^d$. We will make use of the Sobolev spaces $W_p^k(\Omega)$, which are consisting of all functions $f\in L_p(\Omega)$ that have distributional derivatives $D^\alpha f\in L_p(\Omega)$ for all $\alpha\in\mathbb{Z}_+^d$ with $|\alpha|_1 \le k$. Associated with these spaces are the (semi-)norms

$$|f|_{W_p^k(\Omega)}^p := \sum_{|\alpha|_1=k} \|D^\alpha f\|_{L_p(\Omega)}^p \quad \text{and} \quad \|f\|_{W_p^k(\Omega)}^p := \sum_{|\alpha|_1\le k} \|D^\alpha f\|_{L_p(\Omega)}^p$$

for $1 \le p < \infty$, with the usual modification for $p = \infty$. For the special case $\Omega = \mathbb{R}^d$ and $p = 2$, we can define the Sobolev space for arbitrary real number $\tau > 0$ via

$$\|f\|_{W_2^\tau(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} (1+|\theta|^2)^\tau |\hat{f}(\theta)|^2 d\theta\right)^{1/2}.$$

3. Sampling inequalities and approximation order

Definition 3.1 (*[12, Definition 1]*)**.** Let V be a normed linear space with the dual V^*. Given two subspaces $W \subset V$ and $Z \subset W^*$, the set Z is called a *norming set* of W if there is a constant $c > 0$ so that

$$\sup_{z\in Z\,\|z\|=1} |z(w)| \ge c\|w\| \quad \text{for all } w\in W.$$

Lemma 3.2. *For $d = 1, 2, \dots$, define r_d by the formulas $r_1 = 2$ and, if $d > 1$, $r_d = 2d(1 + r_{d-1})$. Suppose that Ω is a compact cube with the side length b. Let n be a positive integer and $\alpha\in\mathbb{Z}_+^d$ with $|\alpha|_1 \le n - 1$. Then for every $X = \{x_1,\dots,x_N\} \subset \Omega$*