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Positive solutions for a class of fractional differential equations with integral boundary conditions

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ABSTRACT

The existence of positive solutions for a class of fractional equations involving the Riemann–Liouville fractional derivative with integral boundary conditions is investigated. By means of the monotone iteration method and some inequalities associated with the Green function, we obtain the existence of a positive solution and establish the iterative sequence for approximating the solution.

1. Introduction

The aim of this paper is to investigate the existence of positive solutions for the fractional differential equation with integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s)u(s)ds, \end{cases} \tag{1.1}$$

where $2 < \alpha \leqslant 3$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α which is defined as follows:

$$D_{0+}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}h(s)ds, \quad n = [\alpha] + 1,$$

where Γ denotes the Euler gamma function and $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$, see [1]. Here, by a positive solution to the problem (1.1), we mean a function $u \in C[0, 1]$, which is positive on $(0, 1)$, and satisfies (1.1).

Fractional differential equations have gained much importance and attention due to the fact that they have been proved to be valuable tools in the modeling of many phenomena in engineering and sciences such as physics, mechanics, economics and biology. In recent years, there has been a great deal of research on the questions of existence and/or uniqueness of solutions (or positive solutions) to boundary value problems for fractional-order differential equations. Among them some work is devoted to the solvability of nonlinear fractional differential equations with integral boundary conditions. For details, see,

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[2–20] and the references therein. In particular, Cabada and Hamdi in [2] investigated the existence of positive solutions for the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \lambda \displaystyle\int_0^1 u(s)ds, \end{cases} \tag{1.2}$$

where $2 < \alpha \leqslant 3$, $\lambda \neq \alpha$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative and f is a continuous function. The authors obtained some existence results by Guo–Krasnoselskii's fixed point theorem. Karakostas [3] provided sufficient conditions for the non-existence of solutions of the boundary-value problems (1.2) in the Caputo sense. Zhao et al. [4] studied the fractional boundary value problem (1.1) replaced $q(t)f(t, u(t))$ by $\lambda h(t)f(u(t))$. The authors obtained some existence results of positive solutions when the nonlinear term satisfies different requirements of superlinearity, sublinearity and the parameter lies in some intervals. The non-existence of positive solutions was also studied, but the question of the computational methods of approximating solutions was not treated.

Motivated by the above mentioned work, our purpose in this paper is to show the existence and iteration of a positive solution to the problem (1.1) by using the monotone iterative method. We not only obtain the existence of a positive solution, but also establish an iterative sequence for approximating the solution. The first term of the iterative sequence may be taken to be a constant function or a simple function. The monotone iterative method has been successfully applied to boundary value problems of integer order ordinary differential equations, see, for example, [21–25] and the references cited therein.

2. Several lemmas

In this section, we present several lemmas that are useful to the proof of our main results. For the forthcoming analysis, we need the following assumptions:

$(H1)$ $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and $f(t, 0) \not\equiv 0$ on $[0, 1]$;
$(H2)$ $g : [0, 1] \to [0, \infty)$ with $g \in L^1[0, 1]$ and $\sigma = \int_0^1 s^{\alpha-1}g(s)ds < 1$, $\theta = \int_0^1 s^{\alpha}g(s)ds$;
$(H3)$ $q : [0, 1] \to [0, \infty)$ with $q \in L^1[0, 1]$ and $0 < \int_0^1 (1-s)^{\alpha-1}q(s)ds < \infty$.

In [4], the authors obtained the Green function associated with the problem (1.1). More precisely, the authors proved the following lemma.

Lemma 2.1 (*[4]*). *For any $h \in C[0, 1]$, the unique solution of the boundary value problem*

$$\begin{cases} D_{0+}^{\alpha}u(t) + h(t) = 0, & 0 \leqslant t \leqslant 1, \\ u(0) = u'(0) = 0, & u(1) = \displaystyle\int_0^1 g(s)u(s)ds, \end{cases}$$

is given by

$$u(t) = \int_0^1 G(t, s)h(s)ds, \quad t \in [0, 1],$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s), \quad (t, s) \in [0, 1] \times [0, 1], \tag{2.1}$$

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.2}$$

and

$$G_2(t, s) = \frac{t^{\alpha-1}}{1-\sigma} \int_0^1 G_1(\tau, s)g(\tau)d\tau. \tag{2.3}$$

Obviously, $G(t, s)$ is continuous on the unit square $[0, 1] \times [0, 1]$.

Lemma 2.2 (*[5]*). *The function $G_1(t, s)$ defined by (2.2) has the following properties:*

$$\frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leqslant G_1(t, s) \leqslant \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}, \quad \forall\, t, s \in [0, 1]. \tag{2.4}$$

The following properties of the Green function play an important role in this paper.

Lemma 2.3. *The Green function $G(t, s)$ defined by (2.1) satisfies the inequalities*

$$p(t)G(s) \leqslant G(t, s) \leqslant G(s), \quad \forall\, t, s \in [0, 1], \tag{2.5}$$