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Novel weak form quadrature element method with expanded Chebyshev nodes

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ABSTRACT

Based on the principle of minimum potential energy and the differential quadrature rule, novel weak form quadrature element method is proposed. Different from the existing ones, expanded Chebyshev grid points are used as the element nodes. A simple but general way is proposed to compute the strains at the integration points explicitly by using the differential quadrature rule. For illustration and verification, quadrature bar and beam elements are established. Several examples are given. Numerical results indicate that the proposed quadrature element method allows a longer time step as compared to elements with other nodes and is an accurate and efficient method for structural analysis.

1. Introduction

The enhancement of computational efficiency and accuracy has always been a matter of interest to the research community of computational mechanics [1–15]. A variety of new and efficient methods have been proposed in recent years, such as the differential quadrature method (DQM) [1.2], discrete singular convolution (DSC) [3], harmonic differential quadrature (HDQ) [4,5], the Adomian decomposition method [6], the combined finite difference and differential transformation method [7], spectral finite element method [8,9], and the moving least square-Ritz method (MLS-Ritz) [10]. To simplify the non-linear formulations as well as to improve the efficiency of the DQM, the Hadamard and SJT products of matrices are introduced by Chen et al. [11–13]. The DSC and DQM have been successfully extended to solve problems of thin and thick plates with irregular shapes [14,15].

Based on the differential quadrature rule [1], the quadrature element method (QEM) has been proposed by Striz et al. [16]. Currently there are two kinds of quadrature element method. One, called the strong form QEM, is formulated based on the governing as well as boundary equations [16–18], and the other, called the weak form QEM, is formulated based on the principle of minimum potential energy [19–21]. It seems that the weak form QEM is more flexible than the strong form QEM and thus would find more applications in the area of structural mechanics, since the method is similar to the higher order finite element method and the time domain spectral element method.

The main character of the weak form QEM is that the differential quadrature rule is used in the formulation of the element stiffness matrix. In the formulations reported by Chen et al. [19], the element can be with any kind of nodes, such as uniformly distributed points, Gauss–Lobatto–Chebyshev points and Gauss quadrature points together with two end points. However, the derivations are complicated and thus the number of element nodes should be small, since it requires

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computing the inverse of a full Vandermonde matrix. To remove this deficiency, Gauss–Lobatto–Legendre (GLL) points are used as the element nodes and the GLL quadrature is used to obtain the element stiffness matrix and mass matrix [20,21]; thus the number of element nodes is not limited and the derivatives at nodal points can be calculated explicitly by the available formulations. This reduces the difficulty in calculations of the derivatives for high order elements, the round off error and programming effort. To use the differential quadrature rule, however, the element nodal points should be the same as the quadrature points; thus only GLL points can be used as the element nodes at the present time.

The expanded Chebyshev points are a very good approximation to the Lebesgue-optimal grid points, which allows the longest time step [22,23]. It is pointed out that the sensitivity to round off error, ease of programming, and maximum time step is likely to be far more important in choosing the "best" grid than slight variations in accuracy for dynamic problems [23].

The objective of this article is to propose a novel weak form quadrature element method. Different from the existing ones in [22,23], expanded Chebyshev grid points are adopted as element nodes. For the ease of programming and reducing the round off error, a simple but general way is proposed to compute the derivatives at the integration points explicitly by using the differential quadrature rule. For illustration and verifications, quadrature bar and beam elements are established. Several examples are given. Conclusions are drawn based on the results reported herein.

2. Weak form quadrature element formulations

The weak form quadrature element formulation process of the stiffness and mass matrices is analogous to the ones of the conventional finite element or time domain spectral element. The crucial ideas of the weak form QEM are adopting specific nodes and computing the derivatives at the integration points by using the differential quadrature rule. This may reduce the programming effort and the round off error, since explicit formulations are available to compute the derivatives at the nodal points.

2.1. Element nodes and shape functions

To achieve the longer time step, expanded Chebyshev points are adopted as the element nodes, namely [23],

$$\xi_k = -\frac{\cos\left((2k-1)\,\pi\,/\,(2N)\right)}{\cos\left(\pi\,/\,(2N)\right)} \quad k = 1, 2, \ldots, N;\; \xi \in [-1, 1]\,. \tag{1}$$

Depending on the problems, the shape functions can be either Lagrange interpolation functions or Hermit interpolation function. Without loss of generality, quadrature bar and beam elements are presented herein for simplicity.

2.2. Quadrature bar element

For a weak form N-node quadrature bar element, the displacement within the element is assumed as

$$u(\xi, t) = \sum_{j=1}^{N} l_j(\xi) u(\xi_j, t) = \sum_{j=1}^{N} l_j(\xi) u_j \tag{2}$$

where $u(\xi, t)$ is the axial displacement, t is the time, and the shape functions are Lagrange interpolation functions defined by

$$l_j(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_{j-1})(\xi-\xi_{j+1})\cdots(\xi-\xi_N)}{(\xi_j-\xi_1)(\xi_j-\xi_2)\cdots(\xi_j-\xi_{j-1})(\xi_j-\xi_{j+1})\cdots(\xi_j-\xi_N)} = \prod_{\substack{k=1\\k\neq j}}^{N} \frac{\xi-\xi_k}{\xi_j-\xi_k}. \tag{3}$$

Substituting Eq. (2) into the strain energy and kinematic energy expressions of a bar element results

$$\begin{aligned} U &= \frac{1}{2}\int_0^L EA\left(\sum_{j=1}^{N}\frac{dl_j(x)}{dx}\,u_j(t)\right)^2 dx = \frac{1}{2}\int_{-1}^{1}\frac{2EA}{L}\left(\sum_{j=1}^{N} l_j'(\xi)u_j(t)\right)^2 d\xi \\ &= \frac{1}{2}\{u\}^T[k]\{u\} \end{aligned} \tag{4}$$

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \rho A\left(\sum_{j=1}^{N} l_j(x)\dot{u}_j(t)\right)^2 dx = \frac{1}{2}\int_{-1}^{1}\frac{\rho AL}{2}\left(\sum_{j=1}^{N} l_j(\xi)\,\dot{u}_j(t)\right)^2 d\xi \\ &= \frac{1}{2}\{\dot{u}\}^T[m]\{\dot{u}\} \end{aligned} \tag{5}$$

where E, A, L and ρ are the elasticity modulus, cross sectional area, element length and mass density of the bar material, $[k]$ and $[m]$ are the stiffness matrix and mass matrix, $\{u\}$ and $\{\dot{u}\}$ are nodal displacement vector and nodal velocity vector,