Applied Mathematics Letters 34 (2014) 76–80

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

# Approximating solutions of nonlinear hybrid differential equations

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ARTICLE INFO

*Article history:*
Received 14 January 2014
Received in revised form 2 April 2014
Accepted 3 April 2014
Available online 13 April 2014

*Keywords:*
Hybrid differential equation
Initial value problems
Hybrid fixed point theorem
Approximate solution

ABSTRACT

We prove the existence as well as approximations of the solutions of initial value problems of first order ordinary nonlinear hybrid differential equations. We rely our results on a recent hybrid fixed point theorem of Dhage (2014) in partially ordered normed linear spaces. A realization is also indicated to illustrate the abstract theory developed in the paper.



## 1. Statement of the problem

Given a closed and bounded interval $J = [t_0, t_0 + a]$, of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0, a > 0$, consider the initial value problem (in short IVP) of first order ordinary nonlinear hybrid differential equation, (in short HDE)

$$\left.\begin{aligned} x'(t) &= f(t, x(t)) + g(t, x(t)), \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

for all $t \in J$, where $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a solution of the HDE (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies Eq. (1.1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J$.

The HDE (1.1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. See Krasnoselskii [1], Burton [2], Dhage [3] and the references therein. The HDE (1.1) is a hybrid differential equation with a linear perturbation of first type and can be tackled with the hybrid fixed point theory (cf. Dhage [3]). The existence theorems proved via classical fixed point theorems on the lines of Krasnoselskii [1] require the condition that the nonlinearities involved in (1.1) satisfy strong Lipschitz and compactness type conditions and do not yield any algorithm to determine the numerical solutions. This is the main motivation of the present paper and it is proved that the existence of the solutions may be proved under weaker partially continuity and partially compactness type conditions.

The paper will be organized as follows. In Section 2 we give some preliminaries and the key hybrid fixed point theorem that will be used in subsequent part of the paper. In Section 3 we establish the main existence result and provide an example to illustrate our main result.

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http://dx.doi.org/10.1016/j.aml.2014.04.002
0893-9659/

## 2. Auxiliary results

The HDE (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on $J$. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by $\|x\| = \sup_{t\in J} |x(t)|$ and $x \leq y \iff x(t) \leq y(t)$ for all $t \in J$ respectively. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ has some nice properties w.r.t. the above order relation in it.

Unless otherwise mentioned, throughout this paper that follows, let $E$ denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that $E$ is *regular* if $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in $E$ such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space $E$ may be found in Nieto and Lopez [4] and Heikkilä and Lakshmikantham [5] and the references therein.

We need the following definitions in the sequel.

**Definition 2.1.** A mapping $\mathcal{T} : E \to E$ is called *isotone* or *nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.

**Definition 2.2.** An operator $\mathcal{T}$ on a normed linear space $E$ into itself is called *compact* if $\mathcal{T}(E)$ is a relatively compact subset of $E$. $\mathcal{T}$ is called *totally bounded* if for any bounded subset $S$ of $E$, $\mathcal{T}(S)$ is a relatively compact subset of $E$. If $\mathcal{T}$ is continuous and totally bounded, then it is called *completely continuous* on $E$.

**Definition 2.3** (*Dhage [6]*)**.** A mapping $\mathcal{T} : E \to E$ is called *partially continuous* at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever $x$ is comparable to $a$ and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on $E$ if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on $E$, then it is continuous on every chain $C$ contained in $E$. $\mathcal{T}$ is called *partially bounded* if $T(C)$ is bounded for every chain $C$ in $E$. $\mathcal{T}$ is called *uniformly partially bounded* if all chains $\mathcal{T}(C)$ in $E$ are bounded by a unique constant.

**Definition 2.4** (*Dhage [7,6]*)**.** An operator $\mathcal{T}$ on a partially normed linear space $E$ into itself is called *partially compact* if $\mathcal{T}(C)$ is a relatively compact subset of $E$ for all totally ordered sets or chains $C$ in $E$. $\mathcal{T}$ is called *partially totally bounded* if for any totally ordered and bounded subset $C$ of $E$, $\mathcal{T}(C)$ is a relatively compact subset of $E$. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on $E$.

**Definition 2.5** (*Dhage [7]*)**.** The order relation $\preceq$ and the metric $d$ on a non-empty set $E$ are said to be *compatible* if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in $E$ and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to $x^*$ implies that the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x^*$. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric $d$ defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set $\mathbb{R}$ with usual order relation $\leq$ and the norm defined by the absolute value function has this property. Similarly, the space $C(J, \mathbb{R})$ with usual order relation $\leq$ and the supremum norm $\|\cdot\|$ are compatible.

**Definition 2.6** (*Dhage [6]*)**.** Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called *partially nonlinear $\mathcal{D}$-Lipschitz* if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|)$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant $k$. If $k < 1$, $\mathcal{T}$ is called a partially contraction with contraction constant $k$. Finally, $\mathcal{T}$ is called nonlinear $\mathcal{D}$-contraction if it is a nonlinear $\mathcal{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$.

The following applicable hybrid fixed point theorem is proved in Dhage [6].

**Theorem 2.7** (*Dhage [6]*)**.** *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in $E$ are compatible. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous ad partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$.*

*Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution $x^*$ in $E$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \ldots$, converges monotonically to $x^*$.*

## 3. Main results

We need the following definition in what follows.