Applied Mathematics Letters 30 (2014) 38–43

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A regularization method for solving the radially symmetric backward heat conduction problem

Wei Cheng [a,*], Yun-Jie Ma [b], Chu-Li Fu [c]

[a] College of Science, Henan University of Technology, Zhengzhou 450001, PR China
[b] School of Mathematics and Information Science, Yantai University, Yantai, Shandong 264005, PR China
[c] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

ARTICLE INFO

Article history:
Received 14 August 2013
Received in revised form 17 December 2013
Accepted 17 December 2013

Keywords:
Ill-posed problem
Backward heat equation
Regularization
Error estimate

ABSTRACT

This work is devoted to solving the radially symmetric backward heat conduction problem, starting from the final temperature distribution. The problem is ill-posed: the solution (if it exists) does not depend continuously on the given data. A modified Tikhonov regularization method is proposed for solving this inverse problem. A quite sharp estimate of the error between the approximate solution and the exact solution is obtained with a suitable choice of regularization parameter. A numerical example is presented to verify the efficiency and accuracy of the method.

1. Introduction

The backward heat conduction problem (BHCP) arises in the modeling of heat propagation in thermophysics and the mechanics of continuous media. The determination of the unknown initial temperature from observable scattered final temperature data is a requirement in many real applications. The problem is also referred to as the final boundary value problem. The BHCP is a classical ill-posed problem [1], and special and effective regularization methods are required.

The BHCP has been considered by using different methods in recent decades. For example, Tautenhahn and Schröter [2] approximated a BHCP by an optimal regularization method. Fu et al. [3] and Nama et al. [4] solved a BHCP by the Fourier method. Wang [5] established a Shannon wavelet regularization method for solving a BHCP. Ternat et al. [6] studied a BHCP by Euler and Crank–Nicolson methods. Ma et al. [7] proposed a variational method for solving the BHCP. It is worth mentioning that Cheng and Liu [8] studied the two-dimensional BHCP. However, most analytical and numerical methods were only used for dealing with heat equations with constant coefficients. A few works based on numerical methods have been presented for backward heat equations with variable coefficients, because the difficulties of these problems are more pronounced than those for the constant coefficient case.

The physical model considered here is a ball of radius r_0, and it is considered radially symmetric with a certain surface heat flux distribution holding at zero. The correspondingly mathematical model can be described via the following radially symmetric BHCP:

$$\begin{cases} u_t = u_{rr} + \dfrac{2}{r}u_r, & 0 < r \le r_0,\ 0 < t < T, \\ u(r,T) = \varphi(r), & 0 \le r \le r_0, \\ u_r(r_0,t) = 0, & 0 \le t \le T, \\ u(r,t) \text{ bounded in } r = 0, & 0 \le t < T, \end{cases} \tag{1.1}$$

* Corresponding author. Tel.: +86 037167756901.
E-mail address: chwei670815@163.com (W. Cheng).

0893-9659/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.aml.2013.12.009

where r is the radial coordinate, and $\varphi(r)$ denotes the final temperature history of the ball. We want to recover the temperature distribution $u(\cdot, t)$ for $0 \le t < T$. This problem is ill-posed. Hence, a regularization is needed.

Cheng et al. [9] have approximated the inverse heat conduction problem by a Tikhonov-type regularization method. In this work, we will use a modified Tikhonov regularization method to stabilize the BHCP (1.1). Introducing a rather technical inequality, we not only obtain a Hölder continuity but also get a logarithmic Hölder-type convergence error estimate, and in particular the logarithmic-type convergence estimate at $t = 0$.

In Section 2 the ill-posedness of problem (1.1) is given. In Section 3 a modified Tikhonov regularization method with a quite sharp error estimate is provided. Finally, numerical results are given in Section 4, to verify the efficiency of our proposed method.

2. Ill-posedness

Throughout this work, we denote by $L^2[0, r_0; r^2]$ the Hilbert space of Lebesgue measurable functions f with weight r^2 on $[0, r_0]$. We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the inner and norm on $L^2[0, r_0; r^2]$, respectively, with the norm

$$\|f\| = \left(\int_0^{r_0} r^2 \, |f(r)|^2 \, dr\right)^{1/2}.$$

As a solution of problem (1.1) we understand a function $u(r, t)$ satisfying (1.1) in the classical sense and for every fixed $t \in [0, T]$, the function $u(\cdot, t) \in L^2[0, r_0; r^2]$. In this class of functions, if the solution of problem (1.1) exists, then it must be unique. We assume that $u(r, t)$ is the unique solution of problem (1.1). We can obtain the following lemma.

Lemma 2.1. *If the solution of problem (1.1) exists, then it is given by*

$$u(r, t) = \sum_{n=1}^{\infty} c_n e^{(\theta_n/r_0)^2(T-t)} \, R_n(r), \tag{2.1}$$

where

$$R_n(r) = \frac{r_0 \sin(\theta_n r/r_0)}{\theta_n r}, \qquad c_n = \frac{4\theta_n^3 \int_0^{r_0} r^2 \varphi(r) R_n(r) \, dr}{r_0^3 (2\,\theta_n - \sin(2\,\theta_n))}. \tag{2.2}$$

Proof. Applying separation of variables, we seek a solution of problem (1.1) with the form

$$u(r, t) = \upsilon(t) R(r). \tag{2.3}$$

Substituting (2.3) into (1.1), we obtain that $\upsilon(t)$ satisfies the equation

$$\upsilon'(t) + \lambda \upsilon(t) = 0, \quad 0 < t \le T, \tag{2.4}$$

and $R(r)$ satisfies the following ordinary equation and boundary conditions:

$$R''(r) + (2/r) \, R'(r) + \lambda \, R(r) = 0, \quad 0 < r \le r_0, \tag{2.5}$$

$$R_r(r_0) = 0, \qquad |R(0)| < +\infty, \tag{2.6}$$

where λ is an unknown constant. It is easy to see that the eigenvalue of problem (2.5)–(2.6) $\lambda > 0$. So we have the general solution of Eq. (2.5):

$$R(r) = A_1 \, j_0(r\sqrt{\lambda}) + A_2 \, y_0(r\sqrt{\lambda}), \quad 0 < r \le r_0, \tag{2.7}$$

where $j_0(x)$ and $y_0(x)$ denote the spherical Bessel functions of the first kind and of the second kind, respectively, which are given by

$$j_0(x) = (\sin x)/x, \qquad y_0(x) = -(\cos x)/x.$$

Using the conditions (2.6) and noting that $\lim_{x \to 0} y_0(x) = -\infty$, we have

$$R(r) = A_1 \, (\sin(r\sqrt{\lambda}))/(r\sqrt{\lambda}).$$

By differentiating $R(r)$ with respect to r and combining with (2.6), we obtain

$$r_0\sqrt{\lambda} = \tan(r_0\sqrt{\lambda}). \tag{2.8}$$

We know that the equation $x = \tan x$ has the sequence of roots $\{\theta_n\}_{n=1}^{\infty}$ which satisfies $0 < \theta_1 < \theta_2 < \cdots < \theta_n < \cdots$, $\lim_{n\to\infty} \theta_n = \infty$. So, the eigenvalues of problem (2.5)–(2.6) are

$$\lambda_n = (\theta_n/r_0)^2, \quad n = 1, 2, \ldots,$$

and the corresponding eigenfunctions are

$$R_n(r) = r_0 \sin(\theta_n r/r_0)/(\theta_n r), \quad n = 1, 2, \ldots. \tag{2.9}$$