Applied Mathematics Letters 30 (2014) 51–55

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

# Weak local residuals as smoothness indicators for the shallow water equations

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ARTICLE INFO

*Article history:*
Received 26 October 2013
Received in revised form 12 December 2013
Accepted 12 December 2013

*Keywords:*
Weak local residual
Smoothness indicator
Shock detector
Finite volume method
Shallow water equations
Conservation laws

ABSTRACT

The system of shallow water equations admits infinitely many conservation laws. This paper investigates weak local residuals as smoothness indicators of numerical solutions to the shallow water equations. To get a weak formulation, a test function and integration are introduced into the shallow water equations. We use a finite volume method to solve the shallow water equations numerically. Based on our numerical simulations, the weak local residual of a simple conservation law with a simple test function is identified to be the best as a smoothness indicator.



## 1. Introduction

Weak local residuals as smoothness indicators or shock detectors for conservation laws were proposed by Karni, Kurganov and Petrova [1,2]. They proved that weak local residuals have higher accuracy on smooth regions than nonsmooth regions. This difference in accuracy makes weak local residuals able to detect the smoothness of numerical solutions and the presence of shock waves. Therefore, weak local residuals are good candidates as refinement indicators for adaptive numerical methods used to solve conservation laws.

In this paper we limit our discussion on the shallow water equations without source terms. We investigate weak local residuals of conservation laws relating to the shallow water equations. Note that these equations admit infinitely many conservation laws as described by Whitham [3]. In formulating the weak local residual, we introduce a test function and integration. Our goal is to find which conservation law of the shallow water equations and which test function should be chosen in order to get a reliable smoothness indicator with the cheapest computation possible.

The rest of this paper is organized as follows. Section 2 presents formulations of weak local residuals of conservation laws. Conservation laws admitted by the shallow water equations are provided in Section 3. Following that, Section 4 reports our numerical experiment results on weak local residuals as smoothness indicators of the shallow water equations. Finally, some concluding remarks are drawn in Section 5.

## 2. Weak local residuals

Consider the scalar conservation law with an initial condition

$$\begin{cases} q_t + f(q)_x = 0, & -\infty < x < \infty, \\ q(x,t) = q_0(x), & t = 0. \end{cases} \tag{1}$$

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0893-9659/$ – see front matter 
http://dx.doi.org/10.1016/j.aml.2013.12.007

Here $x$ is a one-dimensional space variable, $t$ is the time variable, $q$ is the conserved quantity, $f$ is the flux function, and $q_0$ is an arbitrary function defined for an initial condition. The weak form of the initial value problem (1) is

$$\int_0^\infty \int_{-\infty}^\infty [q(x,t)T_t(x,t) + f(q(x,t))T_x(x,t)]\,dx\,dt + \int_{-\infty}^\infty q_0(x)T(x,0)\,dx = 0, \tag{2}$$

where $T(x,t)$ is an arbitrary test function having compact support locally.

Following Karni and Kurganov [2], we take uniform grids ($x_j := j\Delta x$, $t^n := n\Delta t$) and let $q_j^n$ be approximate values of $q(x_j, t^n)$ computed by a conservative method. We denote by $q^\Delta(x,t)$ the corresponding piecewise constant approximation,

$$q^\Delta(x,t) := q_j^n \quad \text{if } (x,t) \in [x_{j-1/2}, x_{j+1/2}] \times [t^{n-1/2}, t^{n+1/2}] \tag{3}$$

where $x_{j\pm 1/2} := x_j \pm \Delta x/2$ and $t^{n\pm 1/2} := t^n \pm \Delta t/2$. We construct a test function $T_j^n(x,t) := B_j(x)B^n(t)$, where $B_j(x)$ and $B^n(t)$ are quadratic $B$-splines centered at $x = x_j$ and $t = t^n$ with the support of size $3\Delta x$ and $3\Delta t$. That is,

$$B_j(x) = \begin{cases} \dfrac{1}{2}\left(\dfrac{x - x_{j-3/2}}{\Delta x}\right)^2 & \text{if } x_{j-3/2} \le x \le x_{j-1/2}, \\ \dfrac{3}{4} - \left(\dfrac{x - x_j}{\Delta x}\right)^2 & \text{if } x_{j-1/2} \le x \le x_{j+1/2}, \\ \dfrac{1}{2}\left(\dfrac{x - x_{j+3/2}}{\Delta x}\right)^2 & \text{if } x_{j+1/2} \le x \le x_{j+3/2}, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

and $B^n(t)$ is defined similarly. Then substituting the test function $T_j^n(x,t)$ into (2) leads to a weak form of the local residual

$$E_j^n = -\int_{t^{n-3/2}}^{t^{n+3/2}} \int_{x_{j-3/2}}^{x_{j+3/2}} \left\{ q^\Delta(x,t)\left[T_j^n(x,t)\right]_t + f(q^\Delta(x,t))\left[T_j^n(x,t)\right]_x \right\} dx\,dt, \tag{5}$$

for conservation laws. After a straightforward computation, the weak local residual (5) can then be expressed as

$$\begin{aligned} E_j^n = {} & \frac{\Delta x}{12}\left[q_{j+1}^{n+1} - q_{j+1}^{n-1} + 4\left(q_j^{n+1} - q_j^{n-1}\right) + q_{j-1}^{n+1} - q_{j-1}^{n-1}\right] \\ & + \frac{\Delta t}{12}\left[f(q_{j+1}^{n+1}) - f(q_{j-1}^{n+1}) + 4\left(f(q_{j+1}^n) - f(q_{j-1}^n)\right) + f(q_{j+1}^{n-1}) - f(q_{j-1}^{n-1})\right]. \end{aligned} \tag{6}$$

This taking of quadratic $B$-splines in constructing the test function adapts from the work of Karni, Kurganov and Petrova [1] on conservation laws. We denote by KKP (Karni–Kurganov–Petrova) indicator the weak local residual (6).

We can also choose the localized linear $B$-splines as the test functions $T_{j+1/2}^{n-1/2}(x,t) := B_{j+1/2}(x)B^{n-1/2}(t)$, where $B_{j+1/2}(x)$ and $B^{n-1/2}(t)$ are centered at $x = x_{j+1/2}$ and $t = t^{n-1/2}$ with the support of size $2\Delta x$ and $2\Delta t$. That is,

$$B_{j+1/2}(x) = \begin{cases} \dfrac{x - x_{j-1/2}}{\Delta x} & \text{if } x_{j-1/2} \le x \le x_{j+1/2}, \\ \dfrac{x_{j+3/2} - x}{\Delta x} & \text{if } x_{j+1/2} \le x \le x_{j+3/2}, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

and $B^{n-1/2}(t)$ is defined similarly. This results in a less expensive computation of the weak local residual

$$E_{j+1/2}^{n-1/2} = -\int_{t^{n-3/2}}^{t^{n+1/2}} \int_{x_{j-1/2}}^{x_{j+3/2}} \left[ q^\Delta(x,t)\left[T_{j+1/2}^{n-1/2}\right]_t + f(q^\Delta(x,t))\left[T_{j+1/2}^{n-1/2}\right]_x \right. dx\,dt, \tag{8}$$

which can be expressed after a straightforward computation as

$$E_{j+1/2}^{n-1/2} = \frac{\Delta x}{2}\left[q_j^n - q_j^{n-1} + q_{j+1}^n - q_{j+1}^{n-1}\right] + \frac{\Delta t}{2}\left[f\left(q_{j+1}^{n-1}\right) - f\left(q_j^{n-1}\right) + f\left(q_{j+1}^n\right) - f\left(q_j^n\right)\right]. \tag{9}$$

This taking of linear $B$-splines in constructing the test function adapts from the work of Constantin and Kurganov [4] on conservation laws. We denote by CK (Constantin–Kurganov) indicator the weak local residual (9).

We restate that in (2), $T(x,t)$ is a locally supported test function. We see from the formulations of CK and KKP indicators, that CK indicator is simpler and cheaper to compute. The CK indicator is constructed from localized linear $B$-splines with the support of size $2\Delta x$ and $2\Delta t$. The KKP indicator is constructed from localized quadratic $B$-splines with the support of size $3\Delta x$ and $3\Delta t$. In theory we can use higher order $B$-splines with larger support size. However, choosing higher order $B$-splines with larger support size as the test function leads to more expensive computations of weak local residuals.