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Solvability of the boundary value problem for stationary magnetohydrodynamic equations under mixed boundary conditions for the magnetic field

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ABSTRACT

The global solvability of the boundary value problem for stationary magnetohydrodynamic equations under the Dirichlet boundary condition for the velocity and mixed boundary conditions for the magnetic field is proved.

1. Introduction. Statement of the boundary value problem

Let Ω be a bounded domain of space $\mathbb{R}^3$ with boundary $\partial\Omega$ consisting of two parts Σ_ν and Σ_τ. We consider the following boundary value problem for stationary magnetohydrodynamic equations of viscous incompressible fluid:

$$\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - æ\,\mathrm{curl}\,\mathbf{H}\times\mathbf{H} = \mathbf{f}, \qquad \mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1}$$

$$\nu_1\mathrm{curl}\,\mathbf{H} - \rho_0^{-1}\mathbf{E} + æ\mathbf{H}\times\mathbf{u} = \nu_1\mathbf{j}, \qquad \mathrm{div}\,\mathbf{H} = 0, \qquad \mathrm{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = 0, \qquad \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}, \qquad \mathbf{E}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}. \tag{3}$$

Here $\mathbf{u}$ is the velocity vector, $\mathbf{H}$ and $\mathbf{E}$ are magnetic and electric fields, respectively, $p = P/\rho_0$, where P is the pressure, $\rho_0 = \text{const}$ is the fluid density, $æ = \mu/\rho_0$, $\nu_1 = 1/\rho_0\sigma = æ\nu_m$, ν and ν_m are constant kinematic and magnetic viscosity coefficients, σ is a constant conductivity, μ is a constant magnetic permeability, $\mathbf{n}$ is the outer normal to $\partial\Omega$, $\mathbf{f}$ is the volume density of external forces, $\mathbf{j}$ is the exterior current density. In the remaining part of the paper we will refer to problem (1)–(3) for given functions $\mathbf{f}$ and $\mathbf{j}$ as problem (1). We note that all the quantities in (1)–(3) are dimensional and their physical dimensions are defined in terms of SI units. Physically the boundary conditions for the electromagnetic field in (3) correspond to the situation when the part Σ_τ of the boundary $\partial\Omega$ is a perfect conductor and other part $\Sigma_\nu \subset \partial\Omega$ is a perfect insulator.

Beginning with the pioneering paper by Solonnikov [1] the solvability of boundary value problems for stationary magnetohydrodynamic equations was studied in a number of papers. Among them we mention [2–7] devoted to the study of the solvability of the boundary value and control problems for stationary MHD equations under the Dirichlet condition for

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the velocity and [8–10] where the MHD equations are considered under mixed boundary conditions for the velocity. As to boundary conditions for magnetic field we note that two main types of conditions were used in cited papers. The first type corresponds to relations $\mathbf{H}\cdot\mathbf{n}=0$ and $\mathbf{E}\times\mathbf{n}=\mathbf{0}$ on $\partial\Omega$ describing the boundary conditions on perfectly conducting boundary (see, e.g., [1,4,5,7,9]). The second type is described by the condition $\mathbf{H}\times\mathbf{n}=\mathbf{0}$ on $\partial\Omega$ [8] corresponding to a perfectly insulating boundary.

It should be noted that when studying flows of conductive fluids in real-life devices the necessity can arise in modelling of flows of conductive fluids in domains with boundaries consisting of parts with different electrical conductivity properties. Mathematical modelling of conductive flows in such type of domains gives rise to studying boundary value problems for MHD equations under mixed boundary conditions for magnetic field. In author's knowledge respective mixed boundary value problems for magnetohydrodynamic equations have not been yet considered in the mathematical literature. At the same time there are some papers in the literature devoted to the study of solvability of corresponding boundary value problems for static Maxwell equations. Among them we mention [11–13] where the solvability of the div–curl system and static Maxwell equations with mixed boundary conditions is studied.

The aim of this paper is to prove the global solvability of mixed boundary value problem (1)–(3) and to establish sufficient conditions to the data which provide the uniqueness of the solution.

2. Function spaces. The preliminary results

Below we will use the Sobolev spaces $H^s(D)$, $s\in\mathbb{R}$, $H^0(D)\equiv L^2(D)$, where D denotes Ω or the boundary $\partial\Omega$. Corresponding spaces of vector-functions are denoted by $H^s(D)^3$ and $L^2(D)^3$. The inner products and norms in the spaces $H^s(\Omega)$ and $H^s(\Omega)^3$ are denoted by $(\cdot,\cdot)_{s,\Omega}$ and $\|\cdot\|_{s,\Omega}$. The inner products and norms in $L^2(\Omega)$ and $L^2(\Omega)^3$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. By $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ we denote the norm and seminorm in $H^1(\Omega)$ or $H^1(\Omega)^3$. For arbitrary Hilbert space H by H^* we denote the dual space of H. As in [12,13] we assume that the following conditions to Ω are satisfied:

(i) Ω is a bounded domain in $\mathbb{R}^3$ and the boundary $\partial\Omega$ is the union of a finite number of disjoint closed C^2 surfaces, each surface having finite surface area;
(ii) Σ_τ is a nonempty open subset of $\partial\Omega$ with $M+1$ disjoint nonempty open components $\{\sigma_0,\sigma_1,\dots,\sigma_M\}$ and there is a positive d_0 such that dist $d(\sigma_i,\sigma_j)\geq d_0>0$ when $i\neq j$ and $M\geq 1$. The boundary of each σ_i is either an empty or $C^{1,1}$ curve. We set $\Sigma_\nu=\partial\Omega\setminus\overline{\Sigma_\tau}$.

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable compactly supported functions in Ω, $H_0^1(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, $V=\{\mathbf{v}\in H_0^1(\Omega)^3:\operatorname{div}\mathbf{v}=0\}$, $H^{-1}(\Omega)^3=(H_0^1(\Omega)^3)^*$, $L_0^2(\Omega)=\{p\in L^2(\Omega):(p,1)=0\}$, $H(\mathrm{curl},\Omega)=\{\mathbf{v}\in L^2(\Omega)^3:\mathrm{curl}\,\mathbf{v}\in L^2(\Omega)^3\}$, $H^1(\Omega,\Sigma_\tau)=\{\varphi\in H^1(\Omega):\varphi|_{\Sigma_\tau}=0\}$, $C_{\Sigma_\tau 0}(\overline{\Omega})^3:=\{\mathbf{v}\in C^0(\overline{\Omega})^3:\mathbf{v}\cdot\mathbf{n}|_{\Sigma_\tau}=0,\ \mathbf{v}\times\mathbf{n}|_{\Sigma_\nu}=\mathbf{0}\}$. In addition to the spaces introduced above we will use the space $H_{DC}(\Omega)=\{\mathbf{v}\in H(\mathrm{curl},\Omega):\operatorname{div}\mathbf{v}\in L^2(\Omega)\}$ equipped with the Hilbert norm

$$\|\mathbf{u}\|_{DC}^2:=\|\mathbf{u}\|^2+\|\operatorname{div}\mathbf{u}\|^2+\|\mathrm{curl}\,\mathbf{u}\|^2. \tag{4}$$

Any vector $\mathbf{v}$ defined on the boundary $\partial\Omega$ (or on a part $\Sigma\subset\partial\Omega$) can be represented as the sum of its normal and tangential components $\mathbf{v}_n$ and $\mathbf{v}_T$: $\mathbf{v}=\mathbf{v}_n+\mathbf{v}_T$. These components are given by $\mathbf{v}_n=(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\equiv v_n\mathbf{n}$ and $\mathbf{v}_T=\mathbf{v}-\mathbf{v}_n\equiv(\mathbf{n}\times\mathbf{v})\times\mathbf{n}$. Here the scalar $v_n=\mathbf{v}\cdot\mathbf{n}$ is the normal component of the vector field $\mathbf{v}$, $\mathbf{v}\times\mathbf{n}$ is the tangential vector which is orthogonal to both the normal $\mathbf{n}$ and to $\mathbf{v}_T$. Obviously, $\mathbf{v}_T=\mathbf{0}$ on Σ if and only if $\mathbf{v}\times\mathbf{n}|_\Sigma=\mathbf{0}$. We will use the following Green's formulae [14]

$$\int_\Omega \mathbf{v}\cdot\mathrm{grad}\,\varphi\,dx+\int_\Omega \operatorname{div}\mathbf{v}\,\varphi\,dx=\int_{\partial\Omega}\mathbf{v}\cdot\mathbf{n}\,\varphi\,d\sigma\quad\forall\mathbf{v}\in H^1(\Omega)^3,\ \varphi\in H^1(\Omega), \tag{5}$$

$$\int_\Omega(\mathbf{v}\cdot\mathrm{curl}\,\mathbf{w}-\mathbf{w}\cdot\mathrm{curl}\,\mathbf{v})dx=\int_{\partial\Omega}(\mathbf{v}\times\mathbf{n})\cdot\mathbf{w}_T d\sigma\quad\forall\mathbf{v},\mathbf{w}\in H^1(\Omega)^3. \tag{6}$$

If $\varphi\in H^1(\Omega,\Sigma_\tau)$ or $\mathbf{w}\in C_{\Sigma_\tau 0}(\overline{\Omega})^3\cap H^1(\Omega)^3$ the right-hand sides of (5) or (6) become $\int_{\Sigma_\nu}\mathbf{v}\cdot\mathbf{n}\,\varphi d\sigma$ or $\int_{\Sigma_\tau}(\mathbf{v}\times\mathbf{n})\cdot\mathbf{w}_T d\sigma$. Based on (5), (6) we say following [12] that the function $\mathbf{v}\in H_{DC}(\Omega)$ satisfies condition $\mathbf{v}\cdot\mathbf{n}=0$ weakly on Σ_ν if

$$\int_\Omega(\mathbf{v}\cdot\mathrm{grad}\,\varphi+\operatorname{div}\mathbf{v}\,\varphi)dx=0\quad\forall\varphi\in H^1(\Omega,\Sigma_\tau).$$

Similarly, we say that function $\mathbf{v}\in H(\mathrm{curl},\Omega)$ satisfies the condition $\mathbf{v}\times\mathbf{n}=\mathbf{0}$ weakly on Σ_τ if

$$\int_\Omega(\mathbf{v}\cdot\mathrm{curl}\,\mathbf{w}-\mathbf{w}\cdot\mathrm{curl}\,\mathbf{v})dx=0\quad\forall\mathbf{w}\in C_{\Sigma_\tau 0}(\overline{\Omega})^3\cap H^1(\Omega)^3.$$

Let $H_{DC\Sigma_\tau}(\Omega)$ be the closure of $C_{\Sigma_\tau 0}(\overline{\Omega})^3\cap H^1(\Omega)^3$ with respect to the norm $\|\cdot\|_{DC}$ in (4). Let

$$\mathcal{H}_{\Sigma_\tau}(\Omega)=\{\mathbf{h}\in L^2(\Omega)^3:\operatorname{div}\mathbf{h}=0,\ \mathrm{curl}\,\mathbf{h}=\mathbf{0}\text{ in }\Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\tau}=0,\ \boldsymbol{h}\times\boldsymbol{n}|_{\Sigma_\nu}=\mathbf{0}\},$$

$$\mathcal{H}_{\Sigma_\nu}(\Omega)=\{\mathbf{h}\in L^2(\Omega)^3:\operatorname{div}\mathbf{h}=0,\ \mathrm{curl}\,\mathbf{h}=\mathbf{0}\text{ in }\Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\nu}=0,\ \boldsymbol{h}\times\boldsymbol{n}|_{\Sigma_\tau}=\mathbf{0}\},$$

$$V_{\Sigma_\tau}(\Omega)=\{\mathbf{v}\in H_{DC\Sigma_\tau}(\Omega):\operatorname{div}\mathbf{v}=0\text{ in }\Omega\}\cap\mathcal{H}_{\Sigma_\tau}(\Omega)^\perp. \tag{7}$$