Applied Mathematics Letters 32 (2014) 48–52

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Operational solution of fractional differential equations

Gabriel Bengochea

Academia de Matemáticas, Universidad Autónoma de la Ciudad de México, México D.F., México

ARTICLE INFO

Article history:
Received 28 November 2013
Received in revised form 11 February 2014
Accepted 24 February 2014
Available online 4 March 2014

Keywords:
Operational calculus
Fractional differential equations
Weyl fractional calculus

ABSTRACT

The main goal of this paper is to solve fractional differential equations by means of an operational calculus. Our calculus is based on a modified shift operator which acts on an abstract space of formal Laurent series. We adopt Weyl's definition of derivatives of fractional order.

1. Introduction

Weyl's definition of the derivative of fractional order α of a function f is given by

$$ {}_tW^{\alpha}_{\infty}f(t) = (-1)^n \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^{\infty} (z-t)^{n-\alpha-1} f(z)dz, \tag{1}$$

where $\mathrm{Re}(\alpha) > 0$ and $t > 0$. The operator ${}_tW^{\alpha}_{\infty}$ acts with respect to t. We will omit the subscripts t and ∞ of the W in (1). For more details, see [1,2]. From definition (1) it is easy to verify that

$$W^{\alpha}e^{-at} = a^{\alpha}e^{-at}, \tag{2}$$

for $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, see [1, 456].

Our goal in this paper is to solve in a purely algebraic way fractional differential equations of the form $q(W^{\alpha})f = g$, where q is a polynomial with complex coefficients, by means of the operational calculus introduced by Bengochea and Verde-Star in [3]. The concepts of equivalence classes and partial fraction decomposition are not used in this theory. We construct a space of formal Laurent series using the abstract objects p_k, $k \in \mathbb{N}$. Also we define the modified shift operator L which acts in the abstract objects as follows: $Lp_0 = 0$ and $Lp_k = p_{k-1}$ for $k \neq 0$. Using the properties of L we can solve equations of the form $q(L)f = g$, where q is a polynomial with complex coefficients, f is unknown and g is a known element in the image of $q(L)$.

Recently, M. Li and W. Zhao applied Mikusiński's operational calculus [4] to solve Abel's type integral equations, see [5], and in [6] they discuss some important aspects of the min-plus algebra related to the convolution product and the asymptotic expression for the identity. M. Khan and M.A. Gondal have constructed a new mechanism for the solution of Abel's type singular integral equations by means of two-step Laplace decomposition algorithm, see [7].

E-mail address: gabriel.bengochea@uacm.edu.mx.

http://dx.doi.org/10.1016/j.aml.2014.02.011
0893-9659/

2. Algebraic setting and preliminary results

In this section we summarize the work introduced in [3]. Let $\{p_k : k \in \mathbb{Z}\}$ be a group with multiplication $p_k p_n = p_{k+n}$, for $n, k \in \mathbb{Z}$. We denote for $\mathcal{F}$ the set of all the formal Laurent series of the form

$$a = \sum_{k \in \mathbb{Z}} a_k p_k,$$

where a_k is a complex number for each $k \in \mathbb{Z}$ and, either, all the a_k are equal to zero, or there exists an integer $w(a)$ such that $a_k = 0$ whenever $k < w(a)$ and $a_{w(a)} \neq 0$. In the first case we write $a = 0$ and define $w(0) = \infty$. The addition and the multiplication by complex numbers in $\mathcal{F}$ are defined in the usual way. For $a = \sum a_k p_k$ and $b = \sum b_k p_k$ in $\mathcal{F}$ we define a multiplication $ab = \sum c_k p_k$, where

$$c_k = \sum_{w(a) \le k \le n - w(b)} a_k b_{n-k}.$$

With this multiplication $\mathcal{F}$ acquires the structure of field, see [3, p. 332]. Observe that the unit element in $\mathcal{F}$ is p_0 and the multiplicative inverse of p_n is p_{-n}. For $x \in \mathbb{C}$, it is true that

$$(p_0 - xp_1) \sum_{n \ge 0} x^n p_n = p_0, \tag{3}$$

and

$$(p_0 - xp_1)^{k+1} p_{-k} \sum_{n \ge k} \binom{n}{k} x^{n-k} p_n = p_0. \tag{4}$$

We use the notation $e_{x,0}$ for $\sum_{n \ge 0} x^n p_n$ and $e_{x,k}$ for $\sum_{n \ge k} \binom{n}{k} x^{n-k} p_n$. The element $e_{x,0}$ is called the geometric series associated with x. From (3) and (4) it follows that $e_{x,0}$ and $p_{-k} e_{x,k}$ are the multiplicative inverses of $p_0 - xp_1$ and $(p_0 - xp_1)^{k+1}$, respectively. It is easy to verify that $p_{-k} e_{x,k} = (e_{x,0})^{k+1}$. We define the linear operator L on $\mathcal{F}$ by $Lp_k = p_{k-1}$, for $k \neq 0$, and $Lp_0 = 0$. This is called the modified left shift. An important property of L is that

$$L^k = p_{-k}(p_0 - P_0 - P_1 - P_2 - \cdots - P_{k-1}), \tag{5}$$

where P_n is the projection on the subspace generated by p_n, this is $P_n a = a_n p_n$. For more details see [3, p. 333].

3. Operational solution of fractional differential equations

From (2) we have that for $\beta = 1/\alpha$

$$W^{\alpha} \mathrm{e}^{-t(1+x)^{\beta}} = (1 + x) \mathrm{e}^{-t(1+x)^{\beta}}, \tag{6}$$

where $\mathrm{Re}((1 + x)^{\beta}) > 0$, $(1 + x)^{\beta} = \mathrm{e}^{\beta \log(1+x)}$, and log denotes the principal branch of the logarithm function. Eq. (6) can be written as

$$(W^{\alpha} - I)\, \mathrm{e}^{-t(1+x)^{\beta}} = x\mathrm{e}^{-t(1+x)^{\beta}}.$$

We define $L = W^{\alpha} - I$, and

$$e_{x,0} = \mathrm{e}^{-t(1+x)^{\beta}}. \tag{7}$$

Then Eq. (6) acquires the form $(L - xI)e_{x,0} = 0$. It is easy to verify that

$$\mathrm{e}^{-t(1+x)^{\beta}} = \sum_{n \ge 0} \sum_{k \ge 0} \binom{k\beta}{n} (-1)^k \frac{t^k}{k!} x^n. \tag{8}$$

From (7), (8), and the fact that $e_{x,0} = \sum_{n \ge 0} x^n p_n$ we obtain

$$p_n = \sum_{k \ge 0} \binom{k\beta}{n} (-1)^k \frac{t^k}{k!}, \quad n \ge 0. \tag{9}$$

The first few p_n are given by

$$p_0 = \sum_{k \ge 0} \binom{k\beta}{0} (-1)^k \frac{t^k}{k!} = u_0(t)\mathrm{e}^{-t},$$

$$p_1 = \sum_{k \ge 0} \binom{k\beta}{1} (-1)^k \frac{t^k}{k!} = u_1(t)\mathrm{e}^{-t},$$