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A note on solutions of the generalized Fisher equation

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ABSTRACT

The generalized Fisher equation is considered. Possible exact solutions of this equation are found by Q-function method. The velocities of traveling waves are determined and analyzed.

The Fisher equation was derived in 1937 and takes the form [1]

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u(x, t)$ is a population density.

Eq. (1) was also introduced and investigated in paper [2] in detail. This equation is often called as the Kolmogorov–Petrovsky–Piscunov equation. Exact solutions of Eq. (1) were first obtained by Ablowitz and Zeppetella in [3]. Later different forms of Eq. (1) were considered many times [4–6].

The aim of this note is to look for exact solutions of the generalized Fisher equation which is also used in biology [7]. This nonlinear evolution equation describes one-dimensional diffusion models for insect and animal dispersal and invasion

$$\frac{\partial u}{\partial t} = u^p(1-u^q) + \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right), \tag{2}$$

where t is time, x is a spatial coordinate, u is a population density, p, q and m are positive parameters. The first term in the right-hand side of Eq. (2) represents the growth of a population. The factor u^m characterizes the diffusion process, which depends on the population density. Insect dispersal is a very important subject which is still not well understood. The above equation is a simple one but even so it gives some hints to possible insect dispersal behavior.

Solutions of Eq. (2) have already been found for some values of parameters p, q, m. In the case of $m = 0, p = 1, q = 1$ we obtain Fisher equation (1). For $m = 0$, $p = 1$ and $q = 2$ we obtain the Burgers–Huxley equation

$$\frac{\partial u}{\partial t} = u - u^3 + \frac{\partial^2 u}{\partial x^2}. \tag{3}$$

Solutions of Eq. (3) were found in many papers as well [8–10].

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Let us consider generalized Fisher equation (2) at $m \neq 0$ using the traveling wave solutions. Assuming in (2)

$$u(x, t) = y(z), \qquad z = kx - \omega t, \quad (k \neq 0) \tag{4}$$

we have a nonlinear ordinary differential equation with parameters p, q and m in the form

$$k^2 \frac{d}{dz}\left(y^m \frac{dy}{dz}\right) + y^p - y^{p+q} + \omega \frac{dy}{dz} = 0. \tag{5}$$

In the case of $m \neq 0$ we can use $y^m = v$. Substituting $y = v^{\frac{1}{m}}$ in Eq. (5), we obtain

$$\frac{k^2}{m} v_z^2 + k^2 v v_{zz} - m\, v^{\frac{p+q+m-1}{m}} + m\, v^{\frac{p+m-1}{m}} + \omega v_z = 0. \tag{6}$$

At the present time there are many methods for finding exact solutions of Eq. (6). Let us list the most useful of them: the tanh-expansion method [11–13], the simplest equation method [14–16], the G'/G-expansion method [17,18]. However in this note we use the method of Q-functions [19]. Advantage of this method was discussed in the recent paper [20].

By means of the Q function method, we can look for exact solutions of Eq. (6) in the form of the following power series [19,20]

$$v(z) = \sum_{j=0}^{N} A_j Q^j(z), \qquad Q(z) = \frac{1}{1 + e^{-z-z_0}}, \tag{7}$$

where N is the pole order and z_0 is an arbitrary constant.

One can see that the function $Q(z)$ is solution of the equation

$$Q_z = Q - Q^2. \tag{8}$$

Eq. (8) allows us to obtain v_z and v_{zz} using polynomials of Q.

One can note that two first members of Eq. (6) have the same pole order and the degree of the third member is more than the degree of the fourth member. So, substituting $v \simeq Q^N$ into three first members of Eq. (6) we have at $q \geq 0$ the following equality

$$\left(\frac{p+q-m-1}{m}\right) N = 2N + 2. \tag{9}$$

From Eq. (9) we obtain the equation in the form

$$\left(\frac{p+q-m-1}{m}\right) = 2 + \frac{2}{N} \tag{10}$$

and we find that the value N should be equal to 1 or 2 in order to have an integer value of $\frac{p+q-m-1}{m}$.

Thus, to look for the solutions of Eq. (6) we can use two formulae

$$v(z) = A_0 + A_1 Q \tag{11}$$

and

$$v(z) = A_0 + A_1 Q + A_2 Q^2. \tag{12}$$

Taking into account $N = 1$ we obtain $p + q = 3m + 1$. In this case we have $p = 3m + 1 - q$ and we get the following equation with respect to $v(z)$

$$\frac{k^2}{m} v_z^2 + k^2 v v_{zz} - m v^4 + m v^{4-\frac{q}{m}} + \omega v_z = 0. \tag{13}$$

We substitute (11) into Eq. (13) and take into consideration different cases for the value q: $q = m$, $q = 2m$, $q = 3m$ and $q = 4m$.

We find that in the case of $q = m$ there are no exact solutions. In the case of $q = 2m$ we obtain the following solutions

$$v(x, t) = \pm 1 \mp 2Q\left(\pm \frac{2m}{\sqrt{2m+1}}\, x \mp \frac{2m}{2m+1}\, t\right). \tag{14}$$

In the case of $q = 3m$ we have not obtained any solutions. But in the case of $q = 4m$ we get the following solutions

$$v(x, t) = \pm 1 \mp 2Q\left(\pm \frac{2m}{\sqrt{2m+1}}\, x \mp \frac{4m(m+1)}{2m+1}\, t\right), \tag{15}$$

$$v(x, t) = \pm i \mp 2iQ\left(\pm \frac{2m}{\sqrt{2m+1}}\, i x \pm \frac{4m(m+1)}{2m+1}\, i t\right). \tag{16}$$