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Soliton solutions to an integrable coupled differential–difference equation

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ABSTRACT

In this paper, we present a new type of coupled differential–difference equation, which comes from the reduction of the discrete analogue of a generalized Toda equation. Using the perturbation method, two-soliton and three-soliton solutions to the coupled system are derived. The N-soliton solution is given in the form of a Pfaffian.

1. Introduction

Integrable generalization of soliton equations is one of the most exciting topics in soliton theory. There are many works on this subject. For example, many coupled extensions of the KdV equation were presented in the literature. There are several approaches to constructing an integrable coupled version of soliton equations. One of them is the bilinear method. Based on bilinear forms, a vector potential KdV equation, a vector Ito equation [1], a vector asymmetrical Nizhnik–Novikov–Veselov equation [2] and a multi-component higher order Ito equation [3] were given. However all of these solitons have a common feature that the phase shift induced by collisions does not depend on the order of collisions. The phase shift plays an important role in determining the type of group acting on the soliton equations [4]. A new type of soliton equation was discussed, in which the phase shift depends on the mutual positions of solitons at the initial time [5,6]. Here the mutual position means the distance between solitons. Nevertheless, the new type of soliton equation exhibits the N-soliton solution that differs from the usual nonlinear superposition of single solitons. The method to construct the new type of soliton equation was also applied to the integrable differential–difference system [7].

A discrete analogue of a generalized Toda equation in the bilinear form is expressed by the following equation [8]

$$[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)] f \bullet f = 0, \tag{1}$$

where Z_i and D_i for $i = 1, 2, 3$, are arbitrary parameter and linear combinations of bilinear operators D_{x_i} respectively. The bilinear difference operator $\exp\left(\delta D_{x_i}\right)$ is defined as [9]

$$\exp\left(\delta D_{x_i}\right) a \bullet b = a\left(x_i + \delta\right) b\left(x_i - \delta\right). \tag{2}$$

Several soliton equations, such as the KdV, modified KdV, Kadomtsev–Petviashvili equation and various types of discrete analogues of soliton equations, can be generated from Eq. (1) by suitably choosing Z_i and D_i and taking the continuum

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limit [8]. In detail, if we choose

$$Z_1 = \frac{1}{\delta} - 1, \qquad Z_2 = \frac{1}{\delta}, \qquad Z_3 = 1 - \frac{2}{\delta}, \tag{3}$$

$$D_1 = D_n + 2\delta D_t, \qquad D_2 = D_n - 2\delta D_t, \qquad D_3 = -D_n. \tag{4}$$

Eq. (1) can be rewritten as

$$\frac{1}{\delta}\sinh(\delta D_t)\left[\frac{1}{\delta}\sinh(\delta D_t)\cosh(D_n) - \frac{1}{2}\sinh(D_n + \delta D_t)\right] f \bullet f = 0. \tag{5}$$

Taking the limit $\delta \to 0$, we obtain the following differential–difference equation

$$D_t\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right) f_n \bullet f_n = 0, \tag{6}$$

where the bilinear differential operator D_t^k is defined as [9]

$$D_t^k a \bullet b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(t) \bullet b(t')\bigg|_{t'=t}. \tag{7}$$

The goal of this paper is to apply the Hirota bilinear approach to conduct the generalization of the differential–difference Eq. (6). Using the perturbation method, we first obtain two- and three-soliton solutions to the new type of integrable coupled equation. Then the N-soliton solution in the form of a Pfaffian is found under Pfaffian identities.

2. Coupled generalization and its multi-soliton solutions

Since Eq. (6) can be represented in the form

$$\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right) f_n \bullet f_{n,t} = 0, \tag{8}$$

we give a natural coupled generalization of (8)

$$\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right) f_n \bullet g_n = 0, \tag{9a}$$

$$D_t\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right) f_n \bullet g_n = 0. \tag{9b}$$

Differentiating Eq. (9a) with respect to t and operating D_t in Eq. (9b) we obtain

$$\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right)(f_{n,t} \bullet g_n + f_n \bullet g_{n,t}) = 0, \tag{10a}$$

$$\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right)(f_{n,t} \bullet g_n - f_n \bullet g_{n,t}) = 0, \tag{10b}$$

that implies

$$\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right) f_n \bullet g_n = 0, \tag{11a}$$

$$\left(D_t\cosh(D_n) - \frac{1}{2}\sinh(D_n)\right) f_n \bullet g_{n,t} = 0. \tag{11b}$$

It is easy to check that Eq. (11) leads to Eq. (6) when $g_n = cf_n$ where c is a nonzero constant. By the dependent variable transformation

$$\frac{g_n}{f_n} = \phi_n, \quad \ln f_n = w_n, \tag{12}$$

we get from (11) the following nonlinear equations

$$\phi_{n+1,t} + \phi_{n-1,t} = (\phi_{n-1} - \phi_{n+1})\left(w_{n+1,t} - w_{n-1,t} - \frac{1}{2}\right), \tag{13a}$$

$$\begin{aligned}\phi_{n+1}w_{n-1,tt} + \phi_{n-1}w_{n+1,tt} \;=\; & (\phi_{n-1}w_{n-1,t} - \phi_{n+1}w_{n+1,t})\left(w_{n+1,t} - w_{n-1,t} - \frac{1}{2}\right) \\ & - \phi_{n+1,t}w_{n+1,t} - \phi_{n-1,t}w_{n-1,t}.\end{aligned} \tag{13b}$$