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Existence of solutions to derivative-dependent, nonlinear singular boundary value problems

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ABSTRACT

This article examines two-point boundary value problems (BVPs) for second-order, singular ordinary differential equations where the right-hand-side of the differential equation may depend on the derivative of the solution. We introduce a method to obtain *a priori* bounds on all potential solutions, including their "derivatives", to the singular BVP under consideration. The approach is based on the application of differential inequalities of singular type. The ideas are then applied to yield new existence results for solutions.

1. Introduction

This work furnishes *a priori* bounds on the solutions to the singular, nonlinear differential equation

$$\frac{1}{p}(p\mathbf{y}')' = q\mathbf{f}(t, \mathbf{y}, p\mathbf{y}'), \quad 0 < t < T \tag{1.1}$$

when subjected to variations of the following boundary conditions

$$-\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \mathbf{c}, \quad (\beta \neq 0); \tag{1.2}$$

$$\mathbf{y}(T) = \mathbf{d}. \tag{1.3}$$

The above boundary conditions feature one Dirichlet condition at one end-point and one Sturm–Liouville condition at the other end-point of the interval of interest. Scalar-valued versions of (1.1)–(1.3) have been considered in papers by Pandey & Verma [1–3] by applying the method of upper and lower solutions to obtain existence of solutions. Alternatively, our approach is based on the construction and application of differential inequalities and Lyapunov-type functions for systems to obtain *a priori* bounds on solutions. The results are then applied to yield novel existence results for solutions to the singular boundary value problem (BVP) (1.1) with (1.2), (1.3).

The study of singular BVPs of type (1.1)–(1.3) is partially motivated by their occurrence in the modeling of various physical phenomena, for example, see [4, O'Regan, Ch. 1].

The recent paper by Fewster-Young & Tisdell [5] analyzed the derivative independent case of (1.1), that is, when $\mathbf{f} = \mathbf{f}(t, \mathbf{y})$. In comparison, the singular BVP associated with the derivative dependent case (1.1) is more challenging, as

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additional bounds on $p\mathbf{y}'$ are required in order to apply topological techniques. This work supplies these *a priori* bounds on $p\mathbf{y}'$, advancing the current state of knowledge in the process.

In (1.1), $\mathbf{f} \in C([0, T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$, that is, $\mathbf{f}$ is a continuous function from $[0, T] \times \mathbb{R}^{2n}$ to $\mathbb{R}^n$. In addition

$$p \in C([0, T]; \mathbb{R}) \cap C^1((0, T); \mathbb{R}) \quad \text{with } p > 0 \text{ on } (0, T) \tag{1.4}$$

$$q \in C((0, T); \mathbb{R}) \quad \text{with } q > 0 \text{ on } (0, T). \tag{1.5}$$

By a solution to (1.1) we mean a function $\mathbf{y} \in C([0, T]; \mathbb{R}^n) \cap C^2((0, T); \mathbb{R}^n)$ with $p\mathbf{y}' \in C([0, T]; \mathbb{R}^n)$ and $\mathbf{y}$ satisfying (1.1) on $(0, T)$.

Our new ideas naturally complement recent advances in the literature, for example, [6,5,7–9] who applied different methods from nonlinear analysis, including fixed-point theory in cones; and Leray–Schauder degree.

For $\mathbf{u} \in \mathbb{R}^n$ we define $\|\mathbf{u}\|$ via $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u}\rangle$, where $\langle \cdot, \cdot \rangle$ is the usual Euclidean dot product of two vectors in $\mathbb{R}^n$.

The Lyapunov-type functions used to form *a priori* bounds on solutions to (1.1)–(1.3) are motivated by the work of: Fewster-Young & Tisdell [5], Hartman [10,11, p. 433]; Mawhin [12]; Rudd and Tisdell [13]; and Erbe, Tisdell and Wong [14]. The type of Lyapunov function used here is the function $r(t) := \|\mathbf{y}(t)\|^2$ where $\mathbf{y} = \mathbf{y}(t)$ is a solution to (1.1). For all $t \in (0, T)$ we have:

$$r'(t) = 2\langle \mathbf{y}(t), \mathbf{y}'(t)\rangle; \quad \text{and} \tag{1.6}$$

$$\begin{aligned}(p(t)r'(t))' &= 2\left[\langle \mathbf{y}(t), p(t)\mathbf{y}'(t)\rangle\right]' = 2\left[\langle \mathbf{y}(t), (p(t)\mathbf{y}'(t))'\rangle + p(t)\|\mathbf{y}'(t)\|^2\right] \\ &= 2\left[\langle \mathbf{y}(t), p(t)q(t)\mathbf{f}(t, \mathbf{y}(t), p(t)\mathbf{y}'(t))\rangle + p(t)\|\mathbf{y}'(t)\|^2\right].\end{aligned} \tag{1.7}$$

The above identities will be needed in the proofs of our main results.

2. Main results

We now present the main results for (1.1) subject to the Dirichlet–Sturm–Liouville boundary conditions (1.2) and (1.3). The following result concerns *a priori* bounds on solutions to (1.1)–(1.3) when $\alpha/\beta \geq 0$ and $\beta \neq 0$. The case when (1.1) is subject to (2.24), (2.25) is essentially the same and is omitted due to space constraints.

Theorem 2.1. *Let $\mathbf{f} \in C([0, T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$. Let $\alpha/\beta \geq 0$ ($\beta \neq 0$), (1.4), (1.5) hold and let*

$$K_1 := \int_0^T \frac{ds}{p(s)} < \infty, \qquad K_2 := \int_0^T p(s)q(s)\, ds < \infty \tag{2.8}$$

with $p^2 q \leq 1$ on $[0, T]$. If there exist non-negative constants V, W such that

$$\|\mathbf{f}(t, \mathbf{u}, \mathbf{v})\| \leq 2V\left(\langle \mathbf{u}, \mathbf{f}(t, \mathbf{u}, \mathbf{v})\rangle + \|\mathbf{v}\|^2\right) + W, \quad \text{for all } (t, \mathbf{u}, \mathbf{v}) \in [0, T] \times \mathbb{R}^{2n} \tag{2.9}$$

then all solutions $\mathbf{y} = \mathbf{y}(t)$ to the singular BVP (1.1)–(1.3) satisfy

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R := \|\mathbf{d}\| + A_1 + V\|\mathbf{d}\|^2 + VK_1^2\|\mathbf{c}\|^2/[\beta(\beta + 2K_1\alpha)] + K_1K_2W, \tag{2.10}$$

$$\text{where } A_1 := K_1 \frac{\|\mathbf{c}\| + |\alpha|(\|\mathbf{d}\| + V\|\mathbf{d}\|^2 + VK_1^2\|\mathbf{c}\|^2/[\beta(\beta + 2K_1\alpha)] + K_1K_2W)}{|\alpha\int_0^T ds/p(s) + \beta|}. \tag{2.11}$$

If, in addition,

$$2V[\|\mathbf{d}\| + A_1 + V\|\mathbf{d}\|^2 + VK_1^2\|\mathbf{c}\|^2/[\beta(\beta + 2K_1\alpha)] + K_1K_2W] < 1 \tag{2.12}$$

then

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq \frac{\|\mathbf{c}\| + R(|\alpha| + \|\mathbf{c}\|) + K_2W|\beta|}{|\beta|(1 - 2V[\|\mathbf{d}\| + A_1 + V\|\mathbf{d}\|^2 + V\tau\|\mathbf{c}\|^2 + K_1K_2W])}.$$

Proof. Let $\mathbf{y} = \mathbf{y}(t)$ be any solution to the singular BVP (1.1)–(1.3). The BVP is equivalent to the following integral equation [4, p. 14, O'Regan]

$$\mathbf{y}(t) = \mathbf{d} - \mathbf{A}\int_t^T \frac{ds}{p(s)} - \int_t^T \frac{1}{p(s)}\int_0^s p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\, dx\, ds, \quad t \in [0, T] \tag{2.13}$$

where

$$\mathbf{A} := \frac{\mathbf{c} + \alpha\left(\mathbf{d} - \int_0^T \frac{1}{p(s)}\int_0^s p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\, dx\, ds\right)}{\alpha\int_0^T \frac{ds}{p(s)} + \beta}.$$