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# Oscillatory behavior of integro-dynamic and integral equations on time scales

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ABSTRACT

By making use of asymptotic properties of nonoscillatory solutions, the oscillation behavior of solutions for the integro-dynamic equation

$$x^{\Delta}(t) = e(t) - \int_0^t k(t,s)f(s,x(s))\Delta s, \quad t \geq 0$$

and the integral equation

$$x(t) = e(t) - \int_0^t k(t,s)f(s,x(s))\Delta s, \quad t \geq 0$$

on time scales is investigated. Easily verifiable sufficient conditions are established for the oscillation of all solutions. The results are new for both continuous and discrete cases. The paper is concluded by an open problem.



## 1. Introduction

The research on oscillation theory for integro-dynamic equations and integral equations is limited due to the lack of techniques available on time scales. In the present work, we introduce a method that could stimulate further research on both integro-dynamic equations and integral equations.

Recall that a time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. For a general background on time scale calculus we refer the reader to the seminal book [1]. Let $\sup \mathbb{T} = \infty$, and denote by $[a,\infty)_{\mathbb{T}}$ a time scale interval, i.e., $[a,\infty)_{\mathbb{T}} = [a,\infty) \cap \mathbb{T}$. By $t \geq a$ we mean $t \in [a,\infty)_{\mathbb{T}}$. For simplicity, assume that $0 \in \mathbb{T}$.

We consider the oscillation problem for the Volterra integro-dynamic equation

$$x^{\Delta}(t) = e(t) - \int_0^t k(t,s)f(s,x(s))\Delta s, \quad t \geq 0 \tag{1}$$

and the Volterra integral equation

$$x(t) = e(t) - \int_0^t k(t,s)f(s,x(s))\Delta s, \quad t \geq 0, \tag{2}$$

where $e : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous, $k(t,\cdot) : [0,\infty)_{\mathbb{T}} \to [0,\infty)$ is rd-continuous for each fixed $t \in \mathbb{T}$, $k(\cdot,s) : [0,\infty)_{\mathbb{T}} \to [0,\infty)$ is rd-continuous for each fixed $s \in \mathbb{T}$, $f(\cdot,x) : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous for each $x \in \mathbb{R}$, and $f(t,\cdot) : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ is continuous for each $t \in \mathbb{T}$.

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Equations of type (1) and (2) arise in many problems of science and engineering such as mathematical models in ecology, population dynamics, spread of epidemics, electric-circuit analysis, semi-conductor devices, mechanics, and plasma physics.

We only consider those solutions of Eqs. (1) and (2) which are nontrivial in the neighborhood of infinity. The term solution henceforth applies to such solutions. As usual, a solution is said to be oscillatory if it is neither eventually positive nor eventually negative. The related equation is called oscillatory if all its solutions are oscillatory.

Although some oscillation theorems for Volterra integro-differential and integral equations in the continuous case can be found in [2–7], the oscillation problem for integral and integro-dynamic equations on time scales is a fairly new topic. To the best of our knowledge, the only study regarding the integro-dynamic equation (1) has been recently carried out in [8]. Therefore, our objective in this paper is to make a further contribution to the subject by studying the oscillation problem for equations of the form (1) and (2).

It is worth mentioning that the results obtained in the present work are new for the continuous case ($\mathbb{T} = \mathbb{R}$), discrete case ($\mathbb{T} = \mathbb{Z}$), $q$-calculus case ($\mathbb{T} = q^{\mathbb{Z}}$), etc. For instance, if $\mathbb{T} = \mathbb{Z}$, then Eqs. (1) and (2) become

$$\Delta x_n = e_n - \sum_{j=0}^{n-1} k(n, j) f(j, x(j)), \quad n \geq 0 \tag{3}$$

and

$$x_n = e_n - \sum_{j=0}^{n-1} k(n, j) f(j, x(j)), \quad n \geq 0, \tag{4}$$

respectively. Obviously, the results of this study are readily available for Eqs. (3) and (4) as special cases.

## 2. The main results

We will employ the following lemma, given in [9, Theorem 3.2.1].

**Lemma 1.** *If $X$ and $Y$ are nonnegative real numbers, then*

$$X^{\lambda} - (1-\lambda)Y^{\lambda} - \lambda X Y^{\lambda-1} \leq 0 \quad \text{for } 0 < \lambda < 1.$$

*The equality holds if and only if $X = Y$.*

Throughout this work, we assume that hypothesis (H) holds.

(H) *There exist rd-continuous functions $a, q, m : [0, \infty)_{\mathbb{T}} \to (0, \infty)$ and a real number $\lambda$, $0 < \lambda \leq 1$, such that*

$$k(t, s) \leq a(t) q(s) \quad \text{for all } t \geq s$$

*and*

$$0 < x f(t, x) \leq m(t)|x|^{\lambda+1} \quad \text{for } x \neq 0 \text{ and } t \geq 0.$$

In what follows, we denote

$$h_{\pm}(t) = e(t) \pm (1-\lambda)\lambda^{\lambda/(1-\lambda)} a(t) \int_0^t p^{\lambda/(\lambda-1)}(s) m^{1/(1-\lambda)}(s) q^{1/(1-\lambda)}(s) \Delta s, \quad 0 < \lambda < 1, \tag{5}$$

where $p : [0, \infty)_{\mathbb{T}} \to (0, \infty)$ is a given rd-continuous function.

### 2.1. Integro-dynamic equations

We first give sufficient conditions under which every nonoscillatory solution of Eq. (1) satisfies

$$x(t) = O(t), \quad t \to \infty.$$

**Theorem 1.** *Let $0 < \lambda < 1$ and (H) hold, and let $h_{\pm}$ be as defined by (5). Assume that*

$$\int_0^{\infty} a(s) \Delta s < \infty \tag{6}$$

*and*

$$\int_0^{\infty} s p(s) \Delta s < \infty. \tag{7}$$

*If*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t h_+(s) \Delta s < \infty, \qquad \liminf_{t\to\infty} \frac{1}{t} \int_0^t h_-(s) \Delta s > -\infty, \tag{8}$$