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On an efficient simultaneous method for finding polynomial zeros

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ABSTRACT

A new iterative method for the simultaneous determination of simple zeros of algebraic polynomials is stated. This method is more efficient compared to the all existing simultaneous methods based on fixed point relations. A very high computational efficiency is obtained using suitable corrections resulting from the Kung–Traub three-step method of low computational complexity. The presented convergence analysis shows that the convergence rate of the basic third order method is increased from 3 to 10 using this special type of corrections and applying $2n$ additional polynomial evaluations per iteration. Some computational aspects and numerical examples are given to demonstrate a very fast convergence and high computational efficiency of the proposed zero-finding method.

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1. Introduction

The aim of this paper is to construct an iterative method for the simultaneous determination of simple polynomial roots with a very high computational efficiency. The proposed method is ranked as the most efficient among existing methods in the class of simultaneous methods for approximating polynomial roots based on fixed point relations. The presented iterative formula relies on the fixed point relation of Gargantini–Henrici type [1]. A high computational efficiency is attained by employing suitable corrections which enable very fast convergence (equal to ten) with minimal computational costs. In fact, these corrections arise from the Kung–Traub three-point method [2].

2. Accelerated methods

Let $f(z) = \prod_{j=1}^{n}(z - \zeta_j)$ be a monic polynomial of degree n with simple real or complex zeros $\zeta_1, \ldots, \zeta_n$ and let

$$u(z) = \frac{f(z)}{f'(z)} = \left[\frac{d}{dz}\log f(z)\right]^{-1} = \left(\sum_{j=1}^{n} \frac{1}{z - \zeta_j}\right)^{-1} \tag{1}$$

be Newton's correction appearing in the quadratically convergent Newton method. To construct an iterative method for the simultaneous inclusion of polynomial zeros, Gargantini and Henrici [1] started from (1) and derived the following fixed point relation

$$\zeta_i = z - \left(\frac{1}{u(z)} - \sum_{j\in \boldsymbol{I}_n\setminus\{i\}} \frac{1}{z - \zeta_j}\right)^{-1} \qquad (i \in \boldsymbol{I}_n := \{1, \ldots, n\}). \tag{2}$$

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Let $z_1, \ldots, z_n$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$. Setting $z = z_i$ and substituting the zeros ζ_j by some approximations z_j^* in (2), the iterative method

$$\hat{z}_i = z_i - \left(\frac{1}{u(z_i)} - \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{1}{z_i - z_j^*} \right)^{-1} \quad (i \in \boldsymbol{I}_n) \tag{3}$$

for the simultaneous determination of all simple zeros of the polynomial f is obtained. The choice $z_j^* = z_j$ in (3) gives the well-known cubically convergent Ehrlich–Aberth method [3,4]

$$\hat{z}_i = z_i - \left(\frac{1}{u(z_i)} - \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{1}{z_i - z_j} \right)^{-1} \quad (i \in \boldsymbol{I}_n). \tag{4}$$

Comparing (2) and (3) it is evident that the better approximations z_j^* give the more accurate approximations $\hat{z}_i$; indeed, if $z_j^* \to \zeta_j$, then $\hat{z}_i \to \zeta_i$. This idea was employed by Nourein in [5] for the construction of the following fourth-order method by using the Newton approximations $z_j^* = z_j - u(z_j)$ in (3):

$$\hat{z}_i = z_i - \left(\frac{1}{u(z_i)} - \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{1}{z_i - z_j + u(z_j)} \right)^{-1} \quad (i \in \boldsymbol{I}_n). \tag{5}$$

In this paper we will prove that further increase of computational efficiency can be achieved by combining a suitable three-point method. More details about multipoint methods may be found in [6,7]. In fact, we construct a tenth-order simultaneous method of the form (3) using $2n$ additional polynomial evaluations. These additional evaluations provide a huge increase of the order of convergence from 3 (method (4)) to the incredible 10.

Let f be a function with an isolated zero ζ and let x_m be its approximation obtained at the mth iterative step. To achieve a very fast convergence of the method (3), we will apply a special case of the Kung–Traub family of multipoint methods of arbitrary order of convergence [2], given through the following three steps:

$$\begin{cases} y_m = x_m - \dfrac{f(x_m)}{f'(x_m)} = x_m - u(x_m), \qquad v_m = y_m - \dfrac{f(x_m)f(y_m)u(x_m)}{\big(f(x_m) - f(y_m)\big)^2}, \\ x_{m+1} = \mathcal{K}(x_m) := v_m - \dfrac{(y_m - v_m)f(v_m)u(x_m)}{\big(f(x_m) - f(v_m)\big)^2} \left[f(y_m) + \dfrac{f(x_m)^2}{f(y_m) - f(v_m)} \right]. \end{cases} \tag{6}$$

For simplicity, the three-point Kung–Traub iteration (6) is denoted as $x_{m+1} = \mathcal{K}(x_m)$.

Now we can construct a new simultaneous method taking the Kung–Traub approximations $z_j^* = \mathcal{K}(z_j)$ (given by (6)) in (3). If $z_1^{(0)}, \ldots, z_n^{(0)}$ are initial approximations to the polynomial zeros $\zeta_1, \ldots, \zeta_n$, then the new simultaneous method is defined by the iterative formula

$$z_i^{(m+1)} = z_i^{(m)} - \left(\frac{1}{u(z_i^{(m)})} - \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{1}{z_i^{(m)} - \mathcal{K}\big(z_j^{(m)}\big)} \right)^{-1}, \quad (i \in \boldsymbol{I}_n,\ m = 0, 1, \ldots). \tag{7}$$

Remark 1. To decrease the total computational cost, before executing an iteration step it is first necessary to calculate all entries $\mathcal{K}\big(z_j^{(m)}\big)$.

3. Convergence analysis

The following theorem deals with the order of convergence of the simultaneous method (7).

Theorem 1. *Assume that initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ are sufficiently close to the distinct zeros $\zeta_1, \ldots, \zeta_n$ of the polynomial f. Then the order of convergence of the simultaneous method (7) is 10.*

Proof. For simplicity, we omit the iteration index m and denote all quantities at the $(m+1)$th iteration with the symbol $\widehat{\ }$. Let us introduce the errors $\varepsilon_j = z_j - \zeta_j$, $\hat{\varepsilon}_j = \hat{z}_j - \zeta_j$, and let

$$z_j^* = \mathcal{K}(z_j), \qquad \lambda_{ij} = z_i - \mathcal{K}(z_j), \qquad \theta_i = \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{\mathcal{K}(z_j) - \zeta_j}{(z_i - \zeta_j)\lambda_{ij}}.$$

Then, starting from (7) and using (1) we obtain

$$\hat{z}_i = z_i - \left(\frac{1}{\varepsilon_i} + \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{1}{z_i - \zeta_j} - \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{1}{\lambda_{ij}} \right)^{-1} = z_i - \frac{\varepsilon_i}{1 - \varepsilon_i \theta_i},$$

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