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On the density of the minimal subspaces generated by discrete linear Hamiltonian systems

Guojing Ren *

School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan, Shandong 250014, PR China

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ABSTRACT

This paper focuses on the density of the minimal subspaces generated by a class of discrete linear Hamiltonian systems. It is shown that the minimal subspace is densely defined if and only if the maximal subspace is an operator; that is, it is single valued. In addition, it is shown that, if the interval on which the systems are defined is bounded from below or above, then the minimal subspace is non-densely defined in any non-trivial case.

1. Introduction

Consider the following discrete linear Hamiltonian system:

$$J\Delta y(t) = (P(t) + \lambda W(t))R(y)(t), \quad t \in \mathcal{I}, \tag{1.1}$$

where $\mathcal{I} := \{t\}_{t=a}^{b}$ is an integral interval, a is a finite integer or $a = -\infty$, and b is a finite integer or $b = +\infty$, $b - a \geq 1$; J is the canonical symplectic matrix, i.e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

and I_n is the $n \times n$ unit matrix; Δ is the forward difference operator, i.e., $\Delta y(t) = y(t+1) - y(t)$; $W(t)$ and $P(t)$ are $2n \times 2n$ Hermitian matrices, and the weight function $W(t) \geq 0$, i.e. $W(t)$ is positive semi-definite for $t \in \mathcal{I}$; the partial right shift operator $R(y)(t) = (u^T(t+1), v^T(t))^T$ with $y(t) = (u^T(t), v^T(t))^T$ and $u(t), v(t) \in \mathbb{C}^n$; and λ is a complex spectral parameter.

Throughout the whole paper, we assume that $W(t)$ is of the block diagonal form,

$$W(t) = \mathrm{diag}\{W_1(t), W_2(t)\},$$

where $W_j(t) \geq 0$ is an $n \times n$ matrix, $j = 1, 2$. Let $P(t)$ be blocked as

$$P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix},$$

* Tel.: +86 0531 88139559.
E-mail addresses: rgjmaths@gmail.com, gjren@sdufe.edu.cn.

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where $A(t)$, $B(t)$, and $C(t)$ are $n \times n$ matrices, $B(t)$ and $C(t)$ are Hermitian matrices, and $A^*(t)$ is the complex conjugate transpose of $A(t)$. Then (1.1) can be written as

$$\Delta u(t) = A(t)u(t+1) + (B(t) + \lambda W_2(t))v(t),$$
$$\Delta v(t) = (C(t) - \lambda W_1(t))u(t+1) - A^*(t)v(t), \quad t \in \mathcal{I}.$$

To ensure the existence and uniqueness of the solution of any initial value problem for (1.1), we always assume that

(A_1) $I_n - A(t)$ is invertible in $\mathcal{I}$.

It is known that (1.1) contains the following formally self-adjoint vector difference equation of order $2m$ [1]:

$$\sum_{j=0}^{m} (-1)^j \Delta^j [p_j(t) \Delta^j z(t-j)] = \lambda w(t) z(t), \quad t \in \mathcal{I}, \tag{1.2}$$

where $w(t)$ and $p_j(t)$, $0 \le j \le m$, are $l \times l$ Hermitian matrices, $w(t) \ge 0$, and $p_m(t)$ is invertible in $\mathcal{I}$.

The spectral theory of self-adjoint operators and self-adjoint extensions of symmetric operators (i.e., densely defined Hermitian operators) in Hilbert spaces has been well developed (see [2,3]). It has been widely applied to study spectral problems of differential operators. For differential system

$$Jy'(x) = (\lambda W(x) + P(x))y(x), \quad x \in \mathcal{I} = [a, b), \tag{1.3}$$

to ensure that the maximal operator is densely defined and single valued, Krall [4] first proposed the following assumption without proof:

$$Jy'(x) - P(x)y(x) = W(x)f(x) \quad \text{and} \quad W(x)y(x) \equiv 0 \quad \text{implies } f(x) \equiv 0 \text{ on } [a, b). \tag{1.4}$$

We mention here that this assumption is reasonable. For details, see Remark 3.1.

In the study of spectral theory of difference expression (1.1) as well as (1.2), it was found that the minimal operator may be a non-densely defined, and the maximal subspace may be not well defined as an operator; that is, it is multi-valued [5]. This fact is ignored in previous literature, including [1,6,8]. More recently, to guarantee the maximal subspace to be single valued, Behncke imposed condition (2.8) in [8], imitating (1.4). But the main result of this paper shows that condition (2.8) in [7] is not satisfied, unless $W(t) \equiv 0$ on $\mathcal{I}$.

Since the adjoint operator is not well defined, the spectral theory of symmetric operators cannot be applied to study spectral problems of these difference operators. So it is necessary and timely for us to study the density of the minimal subspace generated by (1.1) and the conditions for the maximal subspace to be an operator.

Note that the graph $G(T)$ of a linear operator or multi-valued linear operator T in a Hilbert space X is a linear subspace (briefly, subspace) in the product space X^2. A subspace in X^2 is also called a linear relation. In 1961, Arens initiated the study of linear relations [9]. He introduced the concept of adjoint subspace for a subspace in X^2, which is a general subspace whose domain is not required to be dense in X. In addition, he decomposed a closed subspace in X^2 as an operator part and a purely multi-valued part. This decomposition provides a bridge between closed subspaces in X^2 and linear operators in X.

The rest of the paper is organized as follows. In Section 2, some basic concepts about subspaces and some fundamental results about the maximal and minimal subspaces generated by (1.1) are briefly recalled. In Section 3, it is shown that the minimal subspace is densely defined if and only if the maximal subspace is single valued. In addition, it is shown that, if the interval on which the systems are defined is bounded from below or above, the minimal subspace is non-densely defined in any non-trivial case.

2. Preliminaries

In this section, we first recall some basic concepts and useful results about subspaces. For more results about non-densely defined Hermitian operators or Hermitian subspaces, we refer to [9–11] and some of references cited therein.

Let X be a complex Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$, T and S be two subspaces in X^2, and $\lambda \in \mathbb{C}$. Denote

$$D(T) := \{x \in X : (x, f) \in T \text{ for some } f \in X\},$$
$$R(T) := \{f \in X : (x, f) \in T \text{ for some } x \in X\},$$
$$T^* := \{(x, f) \in X^2 : \langle x, g \rangle = \langle f, y \rangle \text{ for all } (y, g) \in T\}.$$

If $T \cap S = \{0\}$, we write

$$T \dotplus S := \{(x + y, f + g) : (x, f) \in T, (y, g) \in S\},$$

which is denoted by $T \oplus S$ in the case that T and S are orthogonal.

Denote

$$T(x) := \{f \in X : (x, f) \in T\}.$$

It can be easily verified that $T(0) = \{0\}$ if and only if T can determine a unique linear operator from $D(T)$ into X whose graph is just T. For convenience, we will identify a linear operator in X with a subspace in X^2 via its graph.