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Lower bounds for the blow-up time in a semilinear parabolic problem involving a variable source

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ABSTRACT

This letter is concerned with the blow-up of the solutions to a semilinear parabolic problem with a reaction given by a variable exponent. Lower bounds for the time of blow-up are derived if the solutions blow up.

1. Introduction

In this letter, we consider the following semilinear parabolic problem with a variable source:

$$\begin{cases} u_t = \Delta u + u^{p(x)}, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n (n \geq 3)$ is a bounded domain with smooth boundary and $u_0(x)$ is the initial value. Moreover, we assume that the function $p(x) : \Omega \to (1, +\infty)$ satisfies

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < +\infty.$$

Problem (1.1) appears in several branches of applied mathematics, for example, it has been used to model chemical reactions, heat transfer and population dynamics. For more information, we refer the interested reader to [1,2] and the references therein.

In [3], the authors proved that there are solutions with blow-up in finite time if and only if $p^+ > 1$. Moreover, they showed that there are functions $p(x)$ and domains Ω such that all solutions of problem (1.1) blow up in finite time. The authors in [4] obtained that the solution of problem (1.1) blows up in finite time when the initial energy is positive.

In this letter, we investigate the lower bound of blow-up time to problem (1.1) in a bounded domain $\Omega \subseteq \mathbb{R}^n (n \geq 3)$.

In the next section, we will find lower bounds for the blow-up time when the blow-up occurs.

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2. A lower bound for the blow-up time

In this section we seek the lower bound for the blow-up time T in some appropriate measure. The idea of the proof of the following theorem is based on the one in [5].

Theorem 2.1. *Let $u(x,t)$ be the nonnegative classical solution of problem (1.1) in a bounded domain $\Omega \subseteq \mathbb{R}^n (n \geq 3)$. Moreover, we assume that the function $p(x)$ satisfies*

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < +\infty.$$

Define

$$\Phi(t) = \int_\Omega u^k \, dx,$$

where k is a parameter restricted by the condition

$$k > \max\left\{2(n-2)(p^+ - 1), 2\right\}. \tag{2.1}$$

If $u(x,t)$ blows up at finite time T, then T is bounded from below by

$$\int_{\Phi(0)}^{+\infty} \frac{d\xi}{k_1 + k_2 \xi^{\frac{3(n-2)}{3n-8}}},$$

where k_1 and k_2 are positive constants which will be determined later.

Proof. First we compute

$$\begin{aligned} \frac{d\Phi}{dt} &= k \int_\Omega u^{k-1} u_t \, dx \\ &= k \int_\Omega u^{k-1} (\Delta u + u^{p(x)}) \, dx \\ &= -\frac{4(k-1)}{k} \int_\Omega \left|\nabla u^{\frac{k}{2}}\right|^2 dx + k \int_\Omega u^{p(x)+k-1} \, dx. \end{aligned} \tag{2.2}$$

For each $t > 0$, we divide Ω into two sets,

$$\Omega_{\{<1\}} = \{x \in \Omega : u(x,t) < 1\}, \qquad \Omega_{\{\geq 1\}} = \{x \in \Omega : u(x,t) \geq 1\}.$$

Now, we have

$$\begin{aligned} \int_\Omega u^{p(x)+k-1} \, dx &= \int_{\Omega_{\{<1\}}} u^{p(x)+k-1} \, dx + \int_{\Omega_{\{\geq 1\}}} u^{p(x)+k-1} \, dx \\ &\leq \int_{\Omega_{\{<1\}}} u^{p^-+k-1} \, dx + \int_{\Omega_{\{\geq 1\}}} u^{p^++k-1} \, dx \\ &\leq \int_\Omega u^{p^-+k-1} \, dx + \int_\Omega u^{p^++k-1} \, dx. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain

$$\frac{d\Phi}{dt} = -\frac{4(k-1)}{k} \int_\Omega \left|\nabla u^{\frac{k}{2}}\right|^2 dx + k \int_\Omega u^{p^-+k-1} \, dx + k \int_\Omega u^{p^++k-1} \, dx. \tag{2.4}$$

By using (2.1), we can apply the Hölder and Young inequalities to get

$$\begin{aligned} \int_\Omega u^{p^-+k-1} \, dx &\leq |\Omega|^{m_1} \left(\int_\Omega u^{\frac{k(2n-3)}{2(n-2)}} \, dx \right)^{m_2} \\ &\leq m_1 |\Omega| + m_2 \int_\Omega u^{\frac{k(2n-3)}{2(n-2)}} \, dx, \end{aligned} \tag{2.5}$$