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Fractional equations with bounded primitive

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ABSTRACT

This article concerns with a class of nonlocal fractional Laplacian problems depending on two real parameters. Our approach is based on variational methods. We establish the existence of three weak solutions via a recent abstract result by Ricceri about nonlocal equations.

1. Introduction

Recently, as observed in [1], a great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. This type of operators arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves.

In this paper, motivated by this large interest, we obtain a multiplicity result for the following nonlocal problem:

$$\begin{cases} -\mathcal{L}_K u = \mu\left(\int_\Omega\left(\int_0^{u(x)} f(t)dt\right)dx - \lambda\right) f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \quad (1)$$

Here and in the sequel, Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$ with $n > 2s$ (where $s \in (0, 1)$), smooth (Lipschitz) boundary $\partial\Omega$ and Lebesgue measure $|\Omega|$, $f : \mathbb{R} \to \mathbb{R}$ is a suitable continuous function with subcritical growth and λ and μ are real parameters.

Moreover, $\mathcal{L}_K$ is the nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n}\Big(u(x+y) + u(x-y) - 2u(x)\Big)K(y)dy, \quad (x \in \mathbb{R}^n)$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function with the properties that:

(k_1) $\gamma K \in L^1(\mathbb{R}^n)$, where $\gamma(x) = \min\{|x|^2, 1\}$;

(k_2) there exists $\beta > 0$ such that

$$K(x) \geq \beta|x|^{-(n+2s)},$$

for any $x \in \mathbb{R}^n \setminus \{0\}$;

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(k_3) $K(x) = K(-x)$, for any $x \in \mathbb{R}^n \setminus \{0\}$.

A typical example for the kernel K is given by $K(x) := |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n.$$

We aim at finding conditions on the data for which problem (1) possesses at least three weak solutions. Our variational approach is realizable checking that the associated energy functional (denoted by J_K) verifies the assumptions requested by a special case (see Theorem 2.1 below) of a recent and general critical point theorem obtained by Ricceri in [2, Theorem 1.6] and thanks to a suitable variational setting developed by Servadei and Valdinoci in [1].

This functional analytical context is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation.

Indeed, the nonlocal analysis that we perform here in order to use Theorem 2.1 is quite general and successfully exploited for other goals in several recent contributions; see [1,3–6] for an elementary introduction to this topic and for a list of related references.

In the nonlocal framework, the simplest example we can deal with is given by the fractional Laplacian, according to the following result:

Theorem 1.1. *Let $s \in (0, 1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero continuous function such that*

$$\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{q-1}} = \lim_{|\xi|\to+\infty} \frac{F(\xi)}{\xi} = 0, \qquad (h^{\star}_{\infty})$$

for some $q \in \left[1, \frac{2n}{n-2s}\right)$, where $F(\xi) := \int_0^{\xi} f(t)\, dt$, for every $\xi \in \mathbb{R}$.

Then, for each μ satisfying

$$\mu > \inf\left\{\left(\int_{\Omega} F(u(x))dx\right)^{-2} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dxdy : u \in A\right\},$$

where

$$A := \left\{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \text{ and } \int_{\Omega} F(u(x))\, dx \neq 0\right\},$$

there exists an open interval

$$\Lambda \subseteq \left(|\Omega| \inf_{\xi\in\mathbb{R}} F(\xi), |\Omega| \sup_{\xi\in\mathbb{R}} F(\xi)\right)$$

such that, for each $\lambda \in \Lambda$, the following equation

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy = \mu\left(\int_{\Omega} F(u(x))dx - \lambda\right) \int_{\Omega} f(u(x))\varphi(x)dx,$$

for every

$$\varphi \in H^s(\mathbb{R}^n) \quad \text{such that } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega,$$

has at least three distinct weak solutions $\{u_j\}_{j=1}^3 \subset H^s(\mathbb{R}^n)$, such that $u_j = 0$ almost everywhere in $\mathbb{R}^n \setminus \Omega$, for every $j \in \{1, 2, 3\}$.

It is worth pointing out that the variational approach to attack such problems is not often easy to perform; indeed due to the presence of the nonlocal term, variational methods do not work when applied to these classes of equations. The plan of the paper is as follows. Section 2 is devoted to our abstract framework and preliminaries. Successively, in Section 3 we give the main result; see Theorem 3.1. Finally, a concrete example of application is presented in Example 3.1.

2. Abstract framework

In this subsection we briefly recall the definition of the functional space X_0, first introduced in [3], and we give some notations. The reader familiar with this topic may skip this section and go directly to the next one. The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$((x, y) \mapsto (g(x) - g(y))\sqrt{K(x-y)}) \in L^2\big((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big),$$