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A remark on the regularity criterion of Boussinesq equations with zero heat conductivity

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ABSTRACT

In this note, we consider the regularity problem under the critical condition to the Boussinesq equations with zero heat conductivity. The Serrin type regularity criteria are established in terms of the critical Besov spaces. This improves a result established in a recent work by Geng and Fan (2012) [6].

1. Introduction and main result

Consider the Boussinesq system with zero heat conductivity in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi = \theta e_3, \quad \text{for } (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \partial_t \theta + u \cdot \nabla \theta = 0, \quad \text{for } (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \nabla \cdot u = 0, \quad \text{for } (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x), \quad \text{for } x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $\theta = \theta(x,t)$ denote the unknown velocity vector field and the scalar temperature, while u_0, θ_0 with $\nabla \cdot u_0 = 0$ in the sense of distribution are given initial data. $e_3 = (0,0,1)^T$. $\pi = \pi(x,t)$ the pressure of fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,\infty)$. The Boussinesq system has important roles in the atmospheric sciences (see e.g. [1]).

Existence and uniqueness theories of solutions to the Boussinesq equations have been studied by many mathematicians and physicists. For the 2D Boussinesq problem, the global in time regularity is well known in [2]. Chae in [3] showed that the 2D Boussinesq system also has a unique smooth global in time solution with zero viscosity. On the other hand, for the 3D Boussinesq problem, Fan and Ozawa [4] and Ishimura and Morimoto [5] proved the following regularity criterion, respectively:

$$\begin{aligned} & u \in L^2\left(0,T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\right), \\ & \nabla u \in L^1\left(0,T; L^\infty(\mathbb{R}^3)\right). \end{aligned} \tag{1.2}$$

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Very recently, Geng and Fan in [6] (see also [7]) established the following criterion for 3D Boussinesq systems:

$$u \in L^{\frac{2}{1-r}}\left(0, T; \dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)\right) \quad \text{with } -1 < r < 1 \text{ and } r \neq 0, \tag{1.3}$$

where $\dot{B}^{s}_{\infty,\infty}$ denotes the homogeneous Besov space. For the regularity problem on other very related models, we refer readers to the investigations [8–11].

Motivated by the result in [6], our aim is to consider the limit case $r = 1$ in (1.3) and we establish a Serrin-type regularity criterion for weak solutions in terms of the velocity in the class $L^{\frac{2}{1-r}}(0, T; \dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3))$, which greatly improves the result in [6]. More precisely, we will prove

Theorem 1.1. *Let $(u_0, \theta_0) \in H^s(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$ and $s > \frac{5}{2}$. Let (u, θ, π) be a local smooth solution to (1.1). If there exists a small positive constant δ such that*

$$\|u\|_{L^\infty\left(0,T,\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right)} \leq \delta, \tag{1.4}$$

then the solution (u, θ, π) to the problem (1.1) remains smooth on $[0, T]$.

Remark 1.1. This result says that the velocity field of the fluids plays a more dominant role than the temperature θ in the regularity theory of the system (1.1). So our theorem is a complement and improvement of the previous results.

Since the critical Morrey–Campanato space $\dot{\mathcal{M}}_{2,3} \subset \dot{B}^{-1}_{\infty,\infty}$ (see e.g. [12,13]), an immediate corollary is as follows.

Corollary 1.2. *Let $(u_0, \theta_0) \in H^s(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$ and $s > \frac{5}{2}$. Let (u, θ, π) be a local smooth solution to (1.1). If there exists a small positive constant δ such that*

$$\|u\|_{L^\infty\left(0,T,\dot{\mathcal{M}}_{2,3}(\mathbb{R}^3)\right)} \leq \delta, \tag{1.5}$$

then the solution (u, θ, π) to the problem (1.1) remains smooth on $[0, T]$.

Remark 1.2. Corollary 1.2 extends previous results and covers the supercritical case [10,14].

Next we recall the definition of the homogeneous Besov space (see e.g. [15,16]). Let $e^{t\Delta}$ denote the heat semi-group defined by

$$e^{t\Delta}f = K_t * f, \qquad K_t(x) = (4\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

for $t > 0$ and $x \in \mathbb{R}^3$, where $*$ means convolution of functions defined on $\mathbb{R}^3$.

We now recall the definition of the homogeneous Besov space with negative indices $\dot{B}^{-\alpha}_{\infty,\infty}$ on $\mathbb{R}^3$ with $\alpha > 0$. It is known [16, p. 192] that $f \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $\dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3)$ if and only if $e^{t\Delta} \in L^\infty$ for all $t > 0$ and $t^{\frac{\alpha}{2}} \left\|e^{t\Delta}f\right\|_\infty \in L^\infty(0, \infty)$. The norm of $\dot{B}^{-\alpha}_{\infty,\infty}$ is defined, up to equivalence, by

$$\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}} = \sup_{t>0}\left(t^{\frac{\alpha}{2}} \left\|e^{t\Delta}f\right\|_\infty\right).$$

Here $\mathcal{S}'$ is the dual of Schwartz space (tempered distribution). The crucial tool in this note is the following lemma which is essentially due to Meyer–Gerard–Oru [17], which plays an important role for the proof of our theorem.

Lemma 1.3. *Let $2 < q < \infty$ and $s = \alpha\left(\frac{q}{2} - 1\right) > 0$. Then there exists a constant C depending only on α and q such that the estimate*

$$\|f\|_{L^q} \leq C \|f\|^{\frac{2}{q}}_{\dot{H}^s} \|f\|^{1-\frac{2}{q}}_{\dot{B}^{-\alpha}_{\infty,\infty}} \tag{1.6}$$

holds for all $f \in \dot{H}^s(\mathbb{R}^3) \cap \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3)$, where $\dot{H}^s$ denotes the homogeneous Sobolev space.

We omit the proof here, the proof can be found in [12,13]. In this paper the letter C denotes a constant which may vary in different cases. Now we are in a position to prove Theorem 1.1.

Proof. Let $T > 0$ be a given fixed time. We assume that u satisfies (1.4). Multiplying the second equation of (1.1) by θ and integrating over $\mathbb{R}^3$, we immediately get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2} = 0.$$