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Exponential relaxation to self-similarity for the superquadratic fragmentation equation

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ABSTRACT

We consider the self-similar fragmentation equation with a superquadratic fragmentation rate and provide a quantitative estimate of the spectral gap.

1. Introduction

The *fragmentation equation*

$$\begin{cases} \partial_t f(t,x) = \mathcal{F} f(t,x), & t \geq 0,\ x > 0, \\ f(0,x) = f_{\text{in}}(x), & x > 0 \end{cases} \tag{1}$$

is a model that describes the time evolution of a population structured with respect to the size x of the individuals.

The key term of the model is the fragmentation operator $\mathcal{F}$, defined as

$$\mathcal{F} f(x) := \int_x^\infty b(y,x) f(y)\,\mathrm{d}y - B(x) f(x). \tag{2}$$

The fragmentation operator quantifies the generation of smaller individuals from a member of the population of size x: the individuals split with a rate $B(x)$ and generate smaller individuals of size $y \in (0,x)$, whose distribution is governed by the kernel $b(x,y)$.

Models involving the fragmentation operator appear in various applications. Among them we can mention crushing of rocks, droplet breakup or combustion [1] which are pure fragmentation phenomena, but also cell division [2], protein polymerization [3] or data transmission protocols on the web [4], for which the fragmentation process occurs together with some "growth" phenomenon.

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In order to ensure the conservation of the total mass of particles which may occur during the fragmentation process, the coefficients $B(x)$ and $b(y,x)$ must be linked through the relation

$$\int_0^y xb(y,x)\,\mathrm{d}x = yB(y). \tag{3}$$

This assumption ensures, at least formally, the mass conservation law

$$\forall t > 0, \quad \int_0^\infty xf(t,x)\,\mathrm{d}x = \int_0^\infty xf_{\mathrm{in}}(x)\,\mathrm{d}x := \rho_{\mathrm{in}}. \tag{4}$$

Moreover, it is well known that $xf(t,x)$ converges to a Dirac mass at $x = 0$ when $t \to +\infty$. Usually, the various contributions that are available in the literature restrict their attention to coefficients which satisfy the homogeneous assumptions (see [5] for instance)

$$B(x) = x^\gamma, \quad \gamma > 0, \quad \text{and} \quad b(y,x) = y^{\gamma-1}p\left(\frac{x}{y}\right), \tag{5}$$

where $\mathrm{d}\mu(z) := p(z)\,\mathrm{d}z$ is a positive measure supported on $[0, 1]$ which satisfies

$$\int_0^1 z\,\mathrm{d}\mu(z) = 1.$$

These hypotheses guarantee that the relation (3) is verified.

From a mathematical point of view, it is convenient to perform the (mass preserving) self-similar change of variable

$$f(t,x) = (1+t)^{2/\gamma} g\left(\frac{1}{\gamma}\log(1+t), (1+t)^{1/\gamma}x\right),$$

or, by writing g in terms of f,

$$g(t,x) := e^{-2t} f(e^{\gamma t} - 1, e^{-t}x).$$

It allows to deduce that $g(t,x)$ satisfies the *self-similar fragmentation equation*

$$\begin{cases} \partial_t g + \partial_x(xg) + g = \gamma \mathcal{F} g, & t \geq 0,\ x > 0, \\ g(0,x) = f_{\mathrm{in}}(x), & x > 0. \end{cases} \tag{6}$$

Eq. (6) belongs to the class of *growth–fragmentation equations* and it admits – unlike Eq. (1) – positive steady-states [6,5,7].

Denote by G the unique positive steady-state of Eq. (6) with normalized mass, i.e. the solution of

$$(xG)' + G = \gamma \mathcal{F} G, \quad G > 0, \qquad \int_0^\infty xG(x)\,\mathrm{d}x = 1.$$

Then it has been proved (see [5,8]) that the solution $g(t,x)$ of the self-similar fragmentation equation (6) converges to $\rho_{\mathrm{in}}G(x)$ when $t \to +\infty$.

Coming back to the fragmentation equation (1), this result implies the convergence of $f(t,x)$ to the self-similar solution $(t,x) \mapsto \rho_{\mathrm{in}}(1+t)^{2/\gamma}G((1+t)^{1/\gamma}x)$ and hence the convergence of $xf(t,x)$ to a Dirac mass δ_0.

In order to obtain more precise quantitative properties of the previous equation, one can wonder about the rate of convergence of $g(t,x)$ to the asymptotic profile $G(x)$. Many recent works are dedicated to this question and prove, under different assumptions and with different techniques, an exponential rate of convergence for growth–fragmentation equations [9,4,10–14,7].

Nevertheless, to our knowledge the only results about the specific case of the self-similar fragmentation equation are those provided by Cáceres, Cañizo and Mischler [10,11]. They prove exponential convergence in the Hilbert space $L^2((x + x^k)\,\mathrm{d}x)$ for a sufficiently large exponent k in [11], and in the Banach space $L^1((x^m + x^M)\,\mathrm{d}x)$ for suitable exponents $1/2 < m < 1 < M < 2$ in [10]. For proving their results, the authors of the aforementioned articles require the measure p to be a bounded function (from above and below) and the power γ of the fragmentation rate to be less than 2.

The current paper aims to obtain a convergence result for superquadratic rates, namely when $\gamma \geq 2$. We obtain exponential convergence to the asymptotic state by working in the weighted Hilbert space $L^2(x\,\mathrm{d}x)$, under the following assumptions:

$$\gamma \geq 2 \quad \text{and} \quad p(z) \equiv 2. \tag{7}$$

The fact that $p(z)$ is a constant means that the distribution of the fragments is uniform: the probability to get a fragment of size x or x' from a particle of size $y > x, x'$ is exactly the same. Then the condition $\int_0^1 zp(z)\,\mathrm{d}z = 1$ imposes this constant to be equal to 2, meaning that the fragmentation is necessarily binary. Our assumption on p is more restrictive than in [10,11], but in return we get a stronger result in the sense that we obtain an estimate of the exponential rate. Now we state the main theorem of this paper.