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Contagion-based distortion risk measures

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A B S T R A C T

We propose a class of distortion measures based on contagion from an external "scenario" variable. The dependence between the scenario and the variable whose risk is measured is modeled with a copula function with horizontal concave sections. Special cases are the perfect dependence copula, which generates expected shortfall, the Marshall–Olkin family and the Placket family. As an application, we evaluate distortion measures bank liabilities with respect to a country risk scenario in the current European debt crisis.

1. Introduction

Distortion risk measures have been proposed to produce conservative risk evaluation in case of particularly negative events. Typically, the approach has been to construct mathematical functions with specific properties to be applied to a probability measure in order to yield a desired distortion of the measure in the tail. The most well known references of this approach are Acerbi [1], Cherny and Madan [2] and Tsukahara [3].

Here we point out a different route to build distortion measures. The idea is that instead of insisting on the desired shape of the distortion, one could select a scenario from which to back out the appropriate distortion function to be applied to the measure. In a sense, we could say that this approach suggests to recover the mathematics of the distortion from an economic argument rather than a requirement on the shape of the probability distribution.

This idea enables to recover well known risk measures from very easy assumptions: for example, evaluating the risk of loss of an asset under the scenario of a huge decrease of the value of the asset itself below a given threshold, would lead us straight to the definition of Expected Shortfall (ES). Since ES could be recovered by applying a specific distortion to a probability measure, this suggests that there could be relationships between scenarios and distortion measures that could be explored further. As a matter of fact, a question arises quite naturally: what about evaluating the risk of the same asset in a scenario of a huge decrease of the reference market, rather than the asset itself? Along the same lines, one could consider the credit risk of a defaultable bond under the scenario of a high probability of a systemic crisis.

Notice that by introducing "contagion" from an outside variable we cast a bridge between the literature on distortion risk measures and that on systemic risk. In fact, all measures proposed in the literature on systemic risk are based on the idea of contagion from an external source (which is typically the systemic crisis) to a risk factor. Concepts that have been proposed to this purpose are Conditional-VaR (CoVar [4]) and Marginal Expected Shortfall (MES [5]). On one side, compared with this literature, the copula function approach that is proposed here is much more general. On the other side, the concepts of CoVaR and MES are more general than that of distortion measures, since they do not impose any concavity requirement of the distortion function. To put it another way, CoVaR and MES measures may fail to qualify as coherent measures. Using copulas we may instead single out the kind of dependence which may induce a proper concave distortion measure.

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The plan of the paper is as follows. In Section 2 we describe contagion-based distortion measures. In Section 3 we propose a set of possible copula functions that may be used to recover proper distortion measures. In Section 4 we present an application to the European sovereign debt crisis.

2. Distortion measures and copula functions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In what follows all random variables are assumed to be continuous and, in particular, we interpret X as a financial position.

Given a set $\mathcal{V}$ of real-valued random variables a *coherent risk measure* (we refer to Artzner et al. [6], that is the seminal paper on this topic) is a function $\rho : \mathcal{V} \to \mathbb{R}$ that is

1. positive: $X \geq 0 \Rightarrow \rho(X) \leq 0$
2. subadditive: $\rho(X+Y) \leq \rho(X) + \rho(Y)$
3. positively homogeneous: $\rho(\lambda X) = \lambda\rho(X)$, for $\lambda > 0$
4. translation invariant: $\rho(A+m) = \rho(X) - m$, $a \in \mathbb{R}$.

A risk measure is called *law invariant* if, for any two random variables X and Y with the same law, we have $\rho(X) = \rho(Y)$ and it is called *comonotonically additive* when $\rho(X+Y) = \rho(X) + \rho(Y)$ if X and Y are comonotone.

A *distortion risk measure* of X is a function of type

$$\rho(X) = -\int_0^1 F_X^{-1}(u)dD(u) = -\int_{\mathbb{R}} xdD(F_X(x)) \tag{1}$$

where D is the distortion function, that is a concave distribution function on [0, 1], with $D(0) = 0$ and $D(1) = 1$ and $F_X^{-1}(u) = \inf\{x : F_X(x) \geq u\}$. These measures are studied in [1] and in [7] and it is proved that any coherent risk measure that is law-invariant and comonotonically additive is of type (1). The concavity requirement of the distortion function D is needed to guarantee properties 1 and 2 of coherent risk measures.

Notice that (1) says that $\rho(X)$ is the expectation of X with respect to the distribution $D \circ F$. Our suggestion is to consider candidates for distortion functions those distributions indexed by $\alpha \in (0, 1]$ of type

$$D_\alpha(u) = \frac{1}{\alpha}C(u, \alpha) \tag{2}$$

where $C(u, \alpha)$ is a function to be defined. The notation of the distortion function reminds of the analysis in [3], even though here we restrict the choice of the distortion function by linking its shape to a copula function.

A *copula function* is a bivariate distribution function on the square $[0, 1]^2$ having uniform margins; more precisely, it is a function $C(u.v) : [0, 1]^2 \to [0, 1]$ such that it is

1. *Grounded*: $C(0, v) = C(u, 0) = 0$.
2. *With uniform marginal*: $C(u, 1) = u$, $C(1, v) = v$.
3. 2-*Increasing*: $C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0$ for $u_1 > u_2$ and $v_1 > v_2$.

The importance of copulas relies on Sklar's Theorem, according to which any joint distribution function $F(x, y)$ can be written as

$$F(x, y) = C(F_X(x), F_Y(y))$$

where C is a copula function and F_X and F_Y are the corresponding marginal distribution functions.

The following result is an immediate application of Lemma 2.1.3 in [8]:

Proposition 2.1. *If D_α is a family of distortion functions for $\alpha \in (0, 1]$, such that $D_1(u) = u$ and for any pair $u_2 \geq u_1$*

$$\alpha(D_\alpha(u_2) - D_\alpha(u_1)) \text{ is an increasing function of } \alpha, \tag{3}$$

then $C(u, \alpha) = \alpha D_\alpha(u)$ is a copula function.

Example 2.1. The Proportional Odds (PO) distortion $D_\alpha(u) = \frac{\alpha u}{1-(1-\alpha)u}$, $\alpha \in (0, 1]$, introduced in [9], defines a copula function, whose horizontal sections are convex. Since in [9] framework the risk measure is referred to the upper tail of losses instead of the lower tail of profit and losses, we may recover the same distortion measure, setting:

$$D_\alpha(u) = \frac{u}{u + (1-u)\alpha}, \quad \alpha \in (0, 1]$$

and the resulting copula is $C(u, \alpha) = \frac{\alpha u}{u+(1-u)\alpha}$ as given in (2.3.4) of [8].