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Risk-neutral valuation of power barrier options

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ABSTRACT

Barrier options are standard exotic options traded in the financial market. These instruments are different from the vanilla options as the payoff of the option depends on whether the underlying asset price reaches a predetermined barrier level, during the life of the option. In this work, we extend the vanilla call barrier options to power call barrier options where the underlying asset price is raised to a constant power, within the standard Black–Scholes framework. It is demonstrated that the pricing of the power barrier options can be obtained from standard barrier options by a transformation which involves the power contract and a adjusted barrier. Numerical results are considered.

1. Introduction

Barrier options are the most widely employed of exotic options. Merton [1] was the first to provide an analytical solution for a vanilla down-and-out barrier option under the assumption of risk-neutrality within the standard [2] framework; see [3] for a review. In addition, [4] extends this by giving complete pricing formulas for all eight types of the standard barrier option. Consequently, [5] discusses standard in and out barrier option, while [6] provides the pricing formulas for the standard barrier options and exotic barrier options. Haug [7] provides a thorough review of standard barrier option and their associated symmetry relations include put-call symmetry.

In this paper we present a generalization of a Merton-type option (given an underlying asset S_t and strike price K). According to [5], the barrier features may be applied to any type of options. We thus apply a barrier to a power option where the underlying asset is raised to a constant factor. Specifically, we present results on pricing options whose payoffs are $\max(S_T^\beta - K, 0)$ with a barrier H where β is a positive constant. We refer to the contract as power barrier options which, to the authors' best knowledge, have not been previously studied in detail in the literature. Wilmott [8] discusses the generalization of standard options to power options; here we specifically point out that a transformation involving a power contract, see [7] for the definition of a power contract, which acts as an 'underlying' brings additional mathematical insights. Power options have the ability to increase leverage in markets where the underlying trades within narrow limits. In addition, power options are closely related to Constant Proportional Portfolio Insurance (CPPI), a major investment strategy; see *e.g.* [9].

CPPI is a portfolio insurance strategy that guarantees the investor's initial investment (for a period typically of 5 years) by dynamically rebalancing the portfolio between a risky asset (e.g. stock) and a risk-free asset. As the price of the risky asset increases a higher proportion is invested in the risky asset and vice versa if the risky asset price falls. The amount of exposure to the risky asset after rebalancing is a constant predetermined multiple of the cushion, the difference between

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the portfolio value and the present value of the guarantee. If the cushion becomes negative the guarantee will not be met at maturity incurring capital loss. [9] shows that options on the CPPI cushion (to hedge this risk) are closely related to power options. In addition, the significance of CPPI products has recently been of interest to regulators given their possible destabilizing effect in falling markets [10].

The power barrier options are composed of a simultaneous generalization of Merton barrier option and a power option of which both have a number of useful applications. For pricing results on power options, see [11]. In addition recent studies have provided combination of options, such as power exchange options [12], Parisian exchange options [13] and compound exchange options [14].

In this paper, we explicitly solve for the closed-form solution of the power barrier options under the assumption of risk neutrality, constant volatility and zero rebate. We also show that there exists a transformation that can be used to obtain the pricing formulas of the power barrier options from the pricing formulas of the vanilla barrier options.

2. Power call barrier options

For a power call option, the pricing formula is given as follows:

$$PC = e^{-r\tau}\int_{\ln\left(\frac{K}{S^\beta}\right)}^{\infty}[S^\beta e^x - K]f(x)\,dx = S^\beta e^{(\beta-1)\left(r+\frac{\beta\sigma^2}{2}\right)\tau}N(d+\beta\sigma\sqrt{\tau}) - Ke^{-r\tau}N(d), \tag{2.1}$$

where r is the constant risk-free rate, σ constant volatility, τ the time to maturity and

$$d = \frac{\ln\left(\frac{S^\beta}{K}\right) + \beta\left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\beta\sqrt{\tau}}. \tag{2.2}$$

It is straightforward to show Eq. (2.1) can be obtained from the standard call option price via the following transformation:

$$S \to S_P = S^\beta e^{(\beta-1)\left(r+\frac{\beta\sigma^2}{2}\right)\tau}, \tag{2.3}$$

$$\sigma \to \hat{\sigma} = \beta\sigma, \tag{2.4}$$

where S_P is a power contract (e.g. [7]). The power contract acts as the 'underlying' for the power option. The power put option, PP, can be obtained via a power put-power call parity relationship:

$$PC + Ke^{-r\tau} = PP + S_P \tag{2.5}$$

where in terms of the transformation, indicated by $(\to)$, is given by

$$C(\to) + Ke^{-r\tau} = P(\to) + S(\to). \tag{2.6}$$

A barrier option can either be a knock in, which is activated upon hitting the barrier, or a knockout which is deactivated upon hitting the barrier. For the latter, if the underlying asset does not reach the barrier during the life of the option, then the holder receives an equivalent payoff of a power option. As for the former, a positive payoff which is equivalent to that of a power option at maturity is only possible when the barrier is hit. Otherwise, the option is worthless. With respect to the underlying, we can either have an up-barrier or a down-barrier where the names are self-explanatory. In this work, for a down-barrier power option, we consider the case for $K > H$, whereas for an up-barrier power option, when $K < H$.

We define $M_T := \max\{S^\beta : t < T\}$, and $m_T := \min\{S^\beta : t < T\}$. Hence the payoffs of down-and-out power call barrier option, $DOPC$, and down-and-in power call barrier option, $DIPC$, are given by

$$DOPC : (S_T^\beta - K)^+\mathbb{1}_{m_T>H}, \tag{2.7}$$

$$DIPC : (S_T^\beta - K)^+\mathbb{1}_{m_T<H}. \tag{2.8}$$

Similarly, the payoffs for an up-and-out power call barrier option, $UOPC$, and an up-and-in power call barrier option, $UIPC$, are given as follows:

$$UOPC : (S_T^\beta - K)^+\mathbb{1}_{M_T<H}, \tag{2.9}$$

$$UIPC : (S_T^\beta - K)^+\mathbb{1}_{M_T>H}. \tag{2.10}$$

The calculation of the expected payoff of the knock in and knockout power barrier options is similar to power options. The only exception is there exists restricted density function for the following case: