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Existence and uniqueness of symmetric positive solutions of 2n-order nonlinear singular boundary value problems☆

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ABSTRACT

By applying an iterative technique, a necessary and sufficient condition is obtained for the existence of symmetric positive solutions of 2n-order nonlinear singular boundary value problems. At the same time, we also show the uniqueness of the symmetric positive solution.

1. Introduction

In this work, we are concerned with the existence of symmetric positive solutions for the 2n-order boundary value problems (BVP)

$$\begin{cases} (-1)^n u^{(2n)}(t) = f(t, u(t)), & t \in (0, 1), \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, & k = 0, 1, 2, \dots, n-1, \end{cases} \tag{1.1}$$

where $n \geq 2$ and $f(t, u)$ may be singular at $u = 0, t = 0$ (and/or $t = 1$). Here, a symmetric positive solution u^* of (1.1) means a solution u^* of (1.1) satisfying

$$u^*(t) = u^*(1-t), \quad t \in [0, 1], \; u^*(t) > 0, \; t \in (0, 1).$$

In recent years, the conditions for the existence and multiplicity of symmetric positive solutions to boundary value problems have been considered in many papers (see [1–6]). In [3], applying the fixed point theorem, Henderson and Thompson obtained the conditions for the existence of at least three symmetric positive solutions to the second-order boundary value problem

$$\begin{cases} y''(t) + f(y) = 0, & t \in [0, 1], \\ y(0) = y(1) = 0. \end{cases}$$

Under the condition that $f(t, u)$ is *non-decreasing* with respect to u, by using the *monotone iterative technique*, Yao [1] proved that the higher-order boundary value problem (1.1) has N symmetric positive solutions and Luo [2] established a necessary and sufficient condition for the existence of symmetric positive solutions to the same problem.

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Under the condition that $f(t, u)$ is *non-increasing* with respect to u, applying the *upper and lower solutions method*, Zhao [4,5] obtained the existence of positive solutions for a class of nonlinear singular boundary value problems (1.1).

Motivated by the work mentioned above, under the condition that $f(t, u)$ is *non-increasing* with respect to u, we, applying the *iterative technique*, give a necessary and sufficient condition for the existence of symmetric positive solutions for $2n$-order nonlinear singular boundary value problem (1.1). Unlike for the case where $f(t, u)$ is non-decreasing with respect to u, we can only construct one *non-monotonic iterative sequence*, which has a non-decreasing subsequence and a non-increasing subsequence.

The main contributions of this work are as follows: (a) the iterative sequence is non-monotonic; (b) the iterative operator is not completely continuous; (c) the search for the iterative initial element is the key point.

To obtain our results, the following conditions will be assumed in this work:

(A_1) $f : (0, 1) \times (0, +\infty) \longrightarrow [0, +\infty)$ is continuous. For $(t, u) \in (0, 1) \times (0, +\infty)$, f is symmetric with respect to t, i.e. f satisfies

$$f(1-t, u) = f(t, u), \quad t \in (0, 1). \tag{1.2}$$

(A_2) For $(t, u) \in (0, 1) \times (0, +\infty)$, f is non-increasing with respect to u and there exists a constant $\lambda \in (0, 1)$ such that for $\sigma \in (0, 1]$,

$$f(t, \sigma u) \le \sigma^{-\lambda} f(t, u). \tag{1.3}$$

From (1.3), it is easy to see that if $\sigma \in [1, +\infty)$, then

$$f(t, \sigma u) \ge \sigma^{-\lambda} f(t, u). \tag{1.4}$$

2. Notation and preliminaries

In this section, we present some material needed in the proof of our main results. Let

$$e(t) = t(1-t), \quad t \in [0, 1], \qquad G(t, s) = \begin{cases} s(1-t), & 0 \le s < t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.1}$$

Obviously for any $t, s \in [0, 1]$, we have $e(t) = G(t, t)$ and

$$e(s)e(t) \le G(t, s) \le G(t, t) = e(t). \tag{2.2}$$

Let E be the Banach space $C[0, 1]$, and define

$$P = \{u \in E : u(0) = u(1) = 0, u(t) > 0 \text{ for } t \in (0, 1), u(t) = u(1-t)$$
$$\text{and for some constant } c \in (0, 1), ce(t) \le u(t) \le c^{-1}e(t) \text{ for } t \in [0, 1]\}. \tag{2.3}$$

By simple computation, we obtain the following Lemma 2.1.

Lemma 2.1. *Let v be integrable on $(0, 1)$; then the BVP*

$$\begin{cases} (-1)^n u^{(2n)}(t) = v(t), & t \in (0, 1), \\ u^{(2k)}(0) = u^{(2k)}(1) = 0, & k = 0, 1, 2, \dots, n-1 \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G_n(t, s) v(s) ds,$$

where

$$G_i(t, s) = \int_0^1 G(t, \tau) G_{i-1}(\tau, s) d\tau, \quad i = 2, \dots, n,$$

$$G_1(t, s) = G(t, s) = \begin{cases} s(1-t), & 0 \le s < t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Remark 2.1. For any $t, s \in [0, 1]$, it is easy to prove that

$$G_n(1-t, 1-s) = G_n(t, s). \tag{2.5}$$

Lemma 2.2. *If $u(t)$ is a symmetric solution of BVP (1.1), then there exists a constant $c \in (0, 1)$ such that*

$$ce(t) \le u(t) \le c^{-1} e(t), \quad t \in [0, 1]. \tag{2.6}$$

The proof is similar to that of Lemma 2.4 in [2].