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Multiple solutions to the nonhomogeneous p-Kirchhoff elliptic equation with concave–convex nonlinearities

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ABSTRACT

In this paper, we study the multiplicity of solutions for the nonhomogeneous p-Kirchhoff elliptic equation

$$-M(\|\nabla u\|_p^p)\Delta_p u = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u + h_3(x), \quad x \in \Omega, \tag{0.1}$$

with zero Dirichlet boundary condition on $\partial\Omega$, where Ω is the complement of a smooth bounded domain D in $\mathbb{R}^N (N \geq 3)$. $\lambda > 0$, $M(s) = a + bs^k$, $a, b > 0$, $k \geq 0$, $h_1(x), h_2(x)$ and $h_3(x)$ are continuous functions which may change sign on Ω. The parameters p, q, r satisfy $1 < q < p(k+1) < r < p^* = \frac{Np}{N-p}$. A new existence result for multiple solutions is obtained by the Mountain Pass Theorem and Ekeland's variational principle.

1. Introduction

In this paper, we are interested in the multiplicity of solutions to the nonhomogeneous p-Kirchhoff elliptic problem of the type

$$\begin{cases} -M(\|\nabla u\|_p^p)\Delta_p u = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u + h_3(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the complement of a smooth bounded domain D in $\mathbb{R}^N$, that is, $\Omega = \mathbb{R}^N \setminus D$. $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$.

Recently, C. Y. Chen et al. in [1] have investigated the multiplicity of solutions to a class of Dirichlet boundary value problems of the type

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\right)\triangle u = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, with $1 < q < 2 < p < 2^* (2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2)$, and the parameters $a, b, \lambda > 0$. The functions $h_1(x), h_2(x) \in C(\overline{\Omega})$ may change sign on Ω. By the Nehari manifold and fibering maps, the existence of multiple positive solutions is established in [1]. For $b = 0$, many authors [2–4] have obtained the existence of at least two positive solutions for $\lambda \in (0, \lambda_0)$. Such nonlocal elliptic problem like (1.2) has received a lot of attention and some interesting results can be found in, for example, [5–7] and the references therein.

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Motivated by these findings, we now extend the analysis to the nonhomogeneous p-Kirchhoff-type equation of (1.1) with the unbounded domain Ω. Replacing the Nehari manifold methods, we will use Ekeland's variational principle and the Mountain Pass Theorem to study the existence of multiple solutions for problem (1.1) with a nonhomogeneous term $h_3(x)$. It seems difficult to get the multiplicity of solutions by Nehari manifold methods.

Throughout this paper, we make the following assumptions.

(H_1) $M(s) = a + bs^k$, $a, b > 0$, $0 \le k < \frac{p}{N-p}$, $\forall s \ge 0$.

(H_2) $h_1 \in L^{q_0}(\Omega) \cap L^\infty(\Omega)$, $h_2 \in L^{r_0}(\Omega) \cap L^\infty(\Omega)$, $h_3 \in L^\mu(\Omega)$ with $q_0 = \frac{p^*}{p^*-q}$, $r_0 = \frac{p^*}{p^*-r}$, $\mu = \frac{p^*}{p^*-1}$. Furthermore, there exists non-empty open domain $\Omega_2 \subset \Omega$ such that $h_2(x) > 0$ in Ω_2.

(H_3) $1 < q < p < p(k+1) < r < p^*$.

Our main result in this paper reads as follows.

Theorem 1. *Let* (H_1)–(H_3) *hold true. Then there exist* $\lambda_0, m_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$, *problem* (1.1) *admits at least two nontrivial weak solutions in* $W_0^{1,p}(\Omega)$ *when* $\|h_3\|_\mu < m_0$.

Remark 1. To the best of our knowledge, it seems that this is the first result about the existence of multiple solutions for the nonhomogeneous p-Kirchhoff elliptic equation with convex–concave nonlinearities.

2. Proof of Theorem 1

In this paper, we let $E = W_0^{1,p}(\Omega)$ be the usual Sobolev spaces with the norm

$$\|u\|_E = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}, \quad 1 \le p < \infty. \tag{2.1}$$

The following Sobolev inequality is well known. There is a constant $S > 0$ such that for every $u \in C_0^\infty(\Omega)$

$$S\left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*} \le \int_\Omega |\nabla u|^p dx; \tag{2.2}$$

see [8,9]. From the standard approximation argument, it is easy to see that (2.2) holds on E.

Definition 1. A function $u \in E$ is said to be a weak solution of problem (1.1) if there holds

$$(a + b\|\nabla u\|_p^{pk}) \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \int_\Omega (\lambda h_1|u|^{q-2}u + h_2|u|^{r-2}u + h_3(x))\varphi dx, \quad \forall \varphi \in E. \tag{2.3}$$

Let $J(u) : E \to \mathbb{R}^1$ be the energy functional of problem (1.1) defined by

$$J(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{m}\|\nabla u\|_p^m - \frac{\lambda}{q}\int_\Omega h_1(x)|u|^q dx - \frac{1}{r}\int_\Omega h_2(x)|u|^r dx - \int_\Omega h_3(x) u dx \tag{2.4}$$

where and in the sequel, we set $m = p(k+1)$. We see $J(u) \in C^1$ and for any $\varphi \in E$, there holds

$$\langle J'(u), \varphi\rangle = (a + b\|\nabla u\|_p^{pk}) \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \int_\Omega (\lambda h_1|u|^{q-2}u + h_2|u|^{r-2}u + h_3)\varphi dx. \tag{2.5}$$

We will make use of the Mountain Pass Theorem in [10] (also see [11]).

Lemma 1 (*Mountain Pass Theorem*). *Let* E *be a real Banach space and* $J \in C^1(E, \mathbb{R})$ *with* $J(0) = 0$. *Suppose* $J(u)$ *satisfies* (*PS*) *condition and*

(A_1) *there are* $\rho, \alpha > 0$ *such that* $J(u) \ge \alpha$ *when* $\|u\|_E = \rho$,
(A_2) *there is* $e \in E$, $\|e\|_E > \rho$ *such that* $J(e) < 0$.

Define

$$\Gamma = \{\gamma \in C^1([0, 1], E) | \gamma(0) = 0, \gamma(1) = e\}. \tag{2.6}$$

Then

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J(\gamma(t)) \ge \alpha \tag{2.7}$$

is a critical value of $J(u)$.