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Reproducing kernel method for singularly perturbed turning point problems having twin boundary layers

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ABSTRACT

A numerical method is proposed for solving singularly perturbed turning point problems exhibiting twin boundary layers based on the reproducing kernel method (RKM). The original problem is reduced to two boundary layers problems and a regular domain problem. The regular domain problem is solved by using the RKM. Two boundary layers problems are treated by combining the method of stretching variable and the RKM. The boundary conditions at transition points are obtained by using the continuity of the approximate solution and its first derivatives at these points. Two numerical examples are provided to illustrate the effectiveness of the present method. The results compared with other methods show that the present method can provide very accurate approximate solutions.

1. Introduction

Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, optimal control, etc. These problems are characterized by the presence of a small parameter that multiplies the highest order derivative, and they are stiff and there exists a boundary or interior layer where the solutions change rapidly.

The numerical treatment of such problems presents some major computational difficulties due to the presence of boundary and interior layers. Also, the singularly perturbed turning point problem is more difficult to handle compared to the non-turning point case. To the best of our knowledge, the discussion on numerical solutions of singularly perturbed turning point problems is rare. Phaneendra1, Reddy and Soujanya [1] proposed a non-iterative numerical integration method on a uniform mesh for dealing with singularly perturbed turning point problems. Rai and Sharma [2–4] discussed the numerical methods for solving singularly perturbed differential-difference equations with turning points. Natesan, Jayakumar and Vigo-Aguiar [5] introduced a parameter uniform numerical method for singularly perturbed turning point problems exhibiting boundary layers. Kadalbajoo, Arora and Gupta [6] developed a collocation method using artificial viscosity for solving the stiff singularly perturbed turning point problem having twin boundary layers.

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, amongst other fields [7–24]. Recently, using the RKM, the authors have discussed two point boundary value problems, nonlocal boundary value problems, partial differential equations and differential-difference equations [10–24]. However, it is very difficult to extend the application of the RKM to singularly perturbed differential equations. Geng [23] developed a method for solving a class of singularly perturbed boundary value problems based on the RKM and a proper transformation. Nevertheless, this method fails to solve singularly perturbed turning point problems.

The objective of this paper is to present a numerical method for solving singularly perturbed turning point problems exhibiting twin boundary layers based on the RKM.

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Consider the following singularly perturbed turning point problems:

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), & -1 < x < 1, \\ u(-1) = \alpha, \quad u(1) = \gamma, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth, and such that (1.1) has a unique solution.

Solution of (1.1) exhibits a layer behavior or turning point behavior depending upon the coefficient $a(x)$. The points of the domain where $a(x) = 0$ are known as turning points. The presence of the turning point results in a boundary or interior layer in the solution of the problem and is more difficult to handle compared to the non-turning point case. In this paper, we consider the case in which the turning point results in twin boundary layers in the solution of the problem.

We consider problem (1.1) with the following assumptions

$$\begin{cases} a(0) = 0, \quad a'(0) < 0, \\ |a(x)| > \alpha > 0, \quad 0 < \tau \le |x| \le 1, \\ b(x) \ge b_0 > 0, \quad x \in [-1, 1], \\ |a'(x)| \ge \dfrac{|a'(0)|}{2}, \quad x \in [-1, 1]. \end{cases} \tag{1.2}$$

Under the above assumptions the given turning point problem possesses a unique solution having twin outflow boundary layers at both the end points [6].

The rest of the paper is organized as follows. In the next section, the numerical technique for (1.1) is introduced. The numerical examples are given in Section 3. Section 4 ends this paper with a brief conclusion.

2. Numerical method

Let $l_\varepsilon = -1 + K\varepsilon$ and $r_\varepsilon = 1 - K\varepsilon$, where K is a positive real number. We divide the given interval $[-1, 1]$ into three nonoverlapping subintervals $[-1, l_\varepsilon]$, $[l_\varepsilon, r_\varepsilon]$ and $[r_\varepsilon, 1]$. The subintervals $[-1, l_\varepsilon]$ and $[r_\varepsilon, 1]$ represent the boundary layers regions, and the subinterval $[l_\varepsilon, r_\varepsilon]$ represents the regular region.

In the boundary layer domains $[-1, l_\varepsilon]$ and $[r_\varepsilon, 1]$, Eq. (1.1) subject to a transition boundary condition at $x = l_\varepsilon$ or $x = r_\varepsilon$ is solved by combining the method of stretching variable and the RKM. The RKM is used to solve Eq. (1.1) with the transition boundary conditions at $x = l_\varepsilon$ and $x = r_\varepsilon$ in the regular domain $[l_\varepsilon, r_\varepsilon]$. In order to obtain the boundary condition at the so-called transition points, continuity of the approximate solution and its first derivatives at these points is used. After solving both the regular and boundary layer problems their solutions are combined to obtain an approximate solution to the original problem over the entire region $[-1, 1]$.

Set $u(l_\varepsilon) = \delta_0$ and $u(r_\varepsilon) = \delta_1$, where δ_0 and δ_1 will be determined approximately at the end of this section.

2.1. Solutions of the regular regions problems

Consider (1.1) in the regular region $[l_\varepsilon, r_\varepsilon]$:

$$\begin{cases} Lu \triangleq \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), & l_\varepsilon < x < r_\varepsilon, \\ u(l_\varepsilon) = \delta_0, \qquad u(r_\varepsilon) = \delta_1. \end{cases} \tag{2.1}$$

Introducing a new unknown function

$$y(x) = u(x) - \phi_1(x), \tag{2.2}$$

where $\phi_1(x) = \gamma_{00} + \gamma_{01}x$ and satisfies $\phi_1(l_\varepsilon) = \delta_0,\ \phi_1(r_\varepsilon) = \delta_1$.

Problem (2.1) with inhomogeneous boundary conditions can be equivalently reduced to the problem of finding a function $y(x)$ satisfying

$$\begin{cases} Ly(x) = F(x), & l_\varepsilon < x < r_\varepsilon, \\ y(l_\varepsilon) = 0, \qquad y(r_\varepsilon) = 0, \end{cases} \tag{2.3}$$

where $F(x) = f(x) - L\phi_1(x)$.

To solve (2.3) using the RKM, first, we construct a reproducing kernel space $W^4[l, r]$.

Definition 2.1. $W^4[l, r] = \{u(x) \mid u'''(x)$ is absolutely continuous, $u^{(4)}(x) \in L^2[l, r]$, $u(l) = 0$, $u(r) = 0\}$. The inner product and norm in $W^4[l, r]$ are given, respectively, by

$$(u(y), v(y))_4 = u(l)v(l) + u'(l)v'(l) + u(r)v(r) + u'(r)v'(r) + \int_l^r u^{(4)}v^{(4)}dy$$

and

$$\|u\|_4 = \sqrt{(u, u)_4}, \quad u, v \in W^4[l, r].$$