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Modeling of self-organized systems interacting with a few individuals: From microscopic to macroscopic dynamics

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ABSTRACT

In nature, self-organized systems such as flocks of birds, schools of fish and herds of sheep have to deal with the presence of external agents such as predators or leaders that modify their internal dynamics. Such situations involve a large number of individuals with their own social behavior interacting with a small number of other individuals acting as external point-source forces. Starting from a microscopic description, we derive a kinetic model using the mean-field limit and introduce a macroscopic model via a suitable hydrodynamic approximation.

1. Introduction

The aim of this study was to present different levels of description for the dynamics of a large group of agents influenced by a small number of external agents. In a biological context, this corresponds to the behavior of a flock or a school of fish attacked by one or more predators, or the movement of a herd of sheep guided by a sheepdog. Such dynamics have been studied in robotic research, whereby engineers tried to control the action of a school of fish by introducing a *fishbot* recognized as a leader [1].

From a modeling viewpoint this involves considering microscopic dynamics as described by classical flocking models, such as those of Cucker and Smale [2] and D'Orsogna et al. [3], with interaction with a set of a few individuals characterized in an approach similar to that used by Colombo and Lécureux-Mercier [4]. Moreover, motivated by a previous analysis [5], we supplemented the classical dynamics of interaction with a *metric* and a *topological interaction rule*.

Following the approach of Carrillo et al. [6], we start from the microscopic dynamics, given by a system of ordinary differential equations (ODEs) and derive two other levels of description: a mesoscopic (or kinetic) level via a *mean-field limit* and a macroscopic level via a suitable *hydrodynamic approximation*. In contrast to the first-order macroscopic models proposed by Colombo and Lécureux-Mercier [4], we obtain second-order models for the corresponding continuum dynamics.

Finally we report some numerical examples for solution of the mean-field model in a series of test cases. The simulations were performed using the fast algorithm recently presented by Albi and Pareschi [7]. Rigorous mathematical results concerning the asymptotic limits just described can be found elsewhere [8].

2. Microscopic model

We are interested in studying a dynamic system composed of N individuals and N^p external agents with the following general structure:

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$$\begin{cases} \dot{x}_i = v_i \quad i = 1, \dots, N \\ \dot{v}_i = \dfrac{1}{N^*} \displaystyle\sum_{j \in \Sigma_N^*(x_i)} F(x_i, v_i, x_j, v_j) + \dfrac{1}{N_p} \sum_{k=1}^{N_p} F^p(x_i, v_i, p_k) \\ \dot{p}_h = \varphi_h(t, \mathbf{p}, \mathcal{A}\rho^N(p_h, t)) \quad h = 1, \dots, N_p, \end{cases} \tag{1}$$

where $(\mathbf{x}, \mathbf{v})_i = (x_i, v_i)$ lives in $\mathbb{R}^{2d}$, $d \geq 1$, $i = 1, \dots, N$ and $(\mathbf{p})_h = p_h \in \mathbb{R}^{nd}$, with $n = 1, 2$, $h = 1, \dots, N_p$ and $N_p \ll N$.

Function F describes the interactions within the swarm and F^p depicts the interaction with each external agent p_h. According to the *three-zone model* [9], F can be decomposed as

$$F(x_i, v_i, x_j, v_j) = H(x_i, x_j)(v_j - v_i) + A(x_i, x_j) + R(x_i, x_j) + S(v_i)v_i, \tag{2}$$

where H is an *alignment* term, A represents *attraction*, R denotes *repulsion* and S is a *self-propulsion–friction* term. The same decomposition holds for F^p if $p_h = (x_h^p, v_h^p)$, i.e. $n = 2$. In a first-order model ($n = 1$), similar interactions can be considered.

We supplement the model with a *topological* rule of interaction. Each agent interacts with only a fixed number of agents of their species:

$$\Sigma_{N^*}(x_i) = \left\{ \text{the } N^* \text{ closest neighbors with respect to } i \right\}.$$

A topological interaction is motivated by recent studies [5,10] suggesting that each individual in a flock modifies its position according to a few individuals directly surrounding it, no matter how close or how far away those agents are. If $N^* = N$, each agent interacts with all the others and the *topological interaction* coincides with the global *metric interaction* [7,11]. Function $\varphi_h : [0, +\infty) \times (\mathbb{R}^{nd})^{N_p} \times \mathbb{R} \longrightarrow \mathbb{R}$ describes the evolution in time of each external individual and depends on the discrete density ρ^N, defined as the empirical measure

$$\rho^N(x, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i(t)).$$

We define $\mathcal{A}$ as a convolution operator $\mathcal{A}\rho^N(x, t) = (\rho^N * \eta)(x, t)$ [4], where η is a smooth kernel with compact support.

2.1. Classical swarming models

Classical swarming models take into account a global interaction between the agents, corresponding in our case to $N^* = N$ neighbors.

Cucker–Smale model. This model describes *alignment dynamics* with

$$H(x_i, x_j)(v_j - v_i) = H(|x_i - x_j|)(v_j - v_i), \tag{3}$$

where $H(|x_i - x_j|)$ is a function that measures the strength of the interaction between individuals i and j and depends on their mutual distance. Under the assumption that closer individuals have more influence than more distant ones, this function is defined as

$$H(r) = \frac{1}{(1 + r^2)^\gamma},$$

where $\gamma \geq 0$ discriminates the behavior of the solution [2,6,11].

D'Orsogna et al. model. This model considers *self-propelling attraction and repulsion dynamics* with

$$A(x_i, x_j) + R(x_i, x_j) + S(v_i)v_i = -\nabla_{x_i} W(|x_j - x_i|) + (\alpha - \beta |v_i|^2) v_i, \tag{4}$$

where $W : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a given potential modeling the short-range repulsion and long-range attraction and α and β are positive parameters. A possible choice is given by the following power law:

$$W(r) = \frac{r^a}{a} - \frac{r^b}{b},$$

where $a > b > 0$ are positive parameters [3].

Both models take into account symmetric interactions between agents, which correspond to the conservation of momentum. Clearly this assumption is not very realistic if we want to model interactions among a group of animals. However, the introduction of other features into the models, such as the notion of a *perception cone* [7,11] or of *relative distance* [12], breaks the interaction symmetry and consequently leads to loss of the conservation of momentum. Note that a *topological interaction* also breaks the symmetry.

3. Kinetic model

To obtain a mesoscopic description of system (1), we use a *mean-field* limit as previously described [6,11]. Since the limit is applied only to the first set of equations describing the evolution of the swarm, we obtain a hybrid model comprising one kinetic equation and a system of ODEs governing the external agents.