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Finite-time Euler singularities: A Lagrangian perspective

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ABSTRACT

We address the question of whether a singularity in a three-dimensional incompressible inviscid fluid flow can occur in finite time. Analytical considerations and numerical simulations suggest high-symmetry flows as promising candidates for finite-time blowup. Utilizing Lagrangian and geometric non-blowup criteria, we present numerical evidence against the formation of a finite-time singularity for the high-symmetry vortex dodecapole initial condition. We use data obtained from high-resolution adaptively refined numerical simulations and inject Lagrangian tracer particles to monitor geometric properties of vortex line segments. We then verify the assumptions made in the analytical non-blowup criteria introduced by Deng et al. [Commun. PDE **31** (2006)] connecting vortex line geometry (curvature, spreading) to velocity increase, to rule out singular behavior.

1. Introduction

The incompressible Euler equations in three dimensions are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

The existence and uniqueness of their solutions for all times are unknown. Together with their prominent counterpart, the incompressible Navier–Stokes equations, these equations have withstood the minds of mathematicians and physicists for centuries. While the latter are regarded as a "Millennium Prize Problem" by the Clay Mathematics Institute [1], the ignorance regarding the existence of global solutions is even larger for the inviscid case: the notion of weak solutions, which has been well established for the Navier–Stokes equations since Leray [2], is unknown for the three-dimensional Euler equations.

As a now classical result, the blowup criterion of Beale et al. [3] (BKM) connects the existence of solutions for the incompressible Euler equations in three dimensions to the critical accumulation of vorticity. Attempts have been made in the past to construct explicit initial conditions for obtaining numerical evidence for or against a finite-time singularity via BKM, with surprisingly inconsistent results [4,5]. The major reason for this ambiguity is the critical dependence on extrapolation, which renders the identification of singular versus near-singular behavior by numerical means next to impossible. Hopes are high that the situation will be less vague when considering geometric analysis of the flow [6–9]. In this letter, we present the application of such geometric criteria to numerical data, to sharpen the distinction between singular and near-singular flow evolution.

The letter is organized as follows: We first review the notion of geometric non-blowup criteria and state the criteria considered and their interpretation. We then briefly describe the computational setup and implementation details for our numerical scheme to obtain adaptively refined data for up to 8192^3 mesh points. Using Lagrangian tracers and diagnostics for vortex line geometry, we analyze the simulation data to conclude a non-blowup for the initial conditions considered. A conclusion and outlook summarize the letter.

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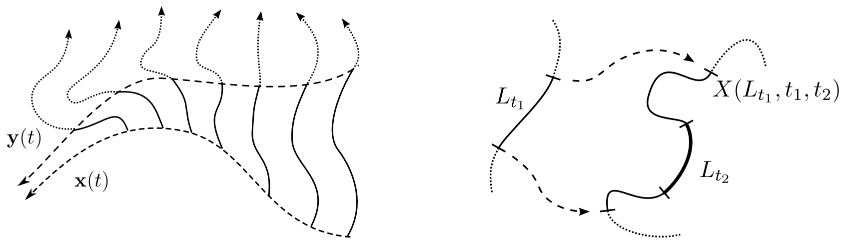

Fig. 1. Left: For the position $\mathbf{x}(t)$ of maximum vorticity, choose $\mathbf{y}(t)$ such that $\left|\int_{\mathbf{x}(t)}^{\mathbf{y}(t)} (\nabla\cdot\boldsymbol{\xi})\,(c(s),t)\mathrm{d}s\right| = C$. For a pointwise singularity, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ must collapse in finite time. Right: Lagrangian evolution of a vortex line segment L_t in the context of Theorem 2. For every $t_2 > t_1$, choose L_{t_2} such that it is a subset of $\mathbf{X}(L_{t_1}, t_1, t_2)$.

2. Geometric non-blowup criteria

Historically, non-blowup criteria for the incompressible Euler equations commonly focus on global features of the flow, such as norms of the velocity or the vorticity fields. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy for overcoming such shortcomings was established by focusing more on geometrical properties and flow structures (see e.g. [6,7,10,11]), such as vortex tubes or vortex lines.

For the Euler equations, this was introduced by Constantin et al. [6]. They improve the BKM criterion by imposing a smoothly directed vorticity field and a bound on the velocity. This criterion takes into account the local structure of the flow and follows the evolution of vortex lines, but the (global) bound on the velocity makes this theorem hard to apply in simulations. A weaker restriction on the velocity field is presented by Cordoba and Fefferman [7]. They consider vortex tubes that are regular in a sense in an $O(1)$ region. With milder assumptions on the surrounding velocity it is shown that the tube cannot reach zero thickness in finite time. Even though the velocity field is no longer required to be uniformly bounded in time, the notion of a "regular tube" of $O(1)$ length is restricting, compared to the experiences with numerical simulations. The blowup criteria considered here [8,9] are inspired by these geometric considerations, but still differ in crucial aspects: the assumptions posed are purely local and restricted to the geometry of a single critical vortex line filament. The assumptions on the velocity do not, in contrast to those of Constantin et al. [6], impose a uniform bound, but they do allow for a blowup of the velocity in finite time, strictly connected in its growth rate to the geometrical evolution of the filament. The vortex line segment itself is not assumed to be of $O(1)$ length (as in [7]) or to be contained in an $O(1)$ region. These aspects in combination render it a promising theorem for being directly tested by means of numerical simulations.

Common to geometric criteria is the notion of vortex lines, defined as integral curves along the vorticity direction field $\boldsymbol{\xi}$. They are transported with the flow, i.e. two points $\mathbf{x}$ and $\mathbf{y}$ on the same vortex line $c(s)$ stay on the same vortex line indefinitely. As a simple consequence of the solenoidality of $\boldsymbol{\omega}$ and the BKM theorem, one gets:

Deng–Hou–Yu Theorem 1. *Let $\mathbf{x}(t)$ be a family of points such that for some $c_0 > 0$ it holds that $|\omega(\mathbf{x}(t),t)| > c_0\Omega(t)$. Assume that for all $t \in [0,T)$ there is another point $\mathbf{y}(t)$ on the same vortex line as $\mathbf{x}(t)$ such that the direction of vorticity $\boldsymbol{\xi}(\mathbf{x},t) = \omega(\mathbf{x},t)/|\omega(\mathbf{x},t)|$ along the vortex line $c(s)$ between $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is well-defined. If we further assume that*

$$\left|\int_{\mathbf{x}(t)}^{\mathbf{y}(t)} (\nabla\cdot\boldsymbol{\xi})\,(c(s),t)\mathrm{d}s\right| \le C \tag{2}$$

for some absolute constant C, and

$$\int_0^T |\omega(\mathbf{y}(t),t)|\mathrm{d}t < \infty, \tag{3}$$

then there will be no blowup up to time T.

This criterion can readily be applied to numerical simulations. On the other hand, the same theorem may be interpreted in a different way to distinguish between a pointwise blowup and the blowup of a complete vortex line segment, as sketched in Fig. 1: At each instance in time, identify the point of maximum vorticity as $\mathbf{x}(t)$. Now define $\mathbf{y}(t)$ such that $|\int_{\mathbf{x}(t)}^{\mathbf{y}(t)} (\nabla\cdot\boldsymbol{\xi})\,(c(s),t)\mathrm{d}s| = C$ for a constant threshold C. If a singularity occurs, then either $\mathbf{y}(t)$ approaches $\mathbf{x}(t)$ (pointwise blowup), or, if the distance between $\mathbf{x}(t)$ and $\mathbf{y}(t)$ stays finite, the complete vortex line segment between $\mathbf{x}(t)$ and $\mathbf{y}(t)$ exhibits critical growth.

Results obtained with this method can be further improved by considering the Lagrangian evolution of vortex line segments L_t in time. The geometric equivalent of the vortex stretching term is the increase in length for a Lagrangian vortex line segment. It is possible to quantify this stretching and establish a sound connection to the vorticity dynamics of the flow.

Denote with $l(t)$ the length of a vortex line segment L_t at time t and define with $\Omega_L(t) := \|\boldsymbol{\omega}(\cdot,t)\|_{L^\infty(L_t)}$ the maximum vorticity on the vortex line segment. Furthermore, let $M(t) := \max(\|\nabla\cdot\boldsymbol{\xi}\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(l_t)})$ be a quantity of vortex line convergence $\nabla\cdot\boldsymbol{\xi}$ and vortex line curvature κ, and define $\lambda(L_t) := M(t)l(t)$. The following then holds (compare Fig. 1):