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Global attractivity of periodic solutions in a higher order difference equation

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ABSTRACT

Consider the following higher order difference equation with periodic coefficients:

$$x_{n+1} = a_n x_n + F(n, x_{n-k}), \quad n = 0, 1, \ldots,$$

where $\{a_n\}$ is a periodic sequence in $(0, 1]$ with period p and $a_n \not\equiv 1$, $F(n, x) : \{0, 1, \ldots\} \times [0, \infty) \to (0, \infty)$ is a continuous function in x and a periodic function in n with period p, and k is a nonnegative integer. We obtain a sufficient condition such that every positive solution of the equation converges to a positive periodic solution. Applications to some difference equations derived from mathematical biology are also given.

1. Introduction

Our aim in this work is to study the global attractivity of periodic solutions of the following higher order nonlinear difference equation:

$$x_{n+1} = a_n x_n + F(n, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.1}$$

where $\{a_n\}$ is a periodic sequence in $(0, 1]$ with period p and $a_n \not\equiv 1$, $F(n, x) : \{0, 1, \ldots\} \times [0, \infty) \to (0, \infty)$ is a continuous function in x and a periodic function in n with period p, and k is a nonnegative integer.

By a solution of Eq. (1.1), we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies Eq. (1.1) for $n \geq 0$. If we let

$$x_{-k}, x_{-k+1}, \ldots, x_0 \tag{1.2}$$

be $k + 1$ given nonnegative numbers with $x_0 > 0$, then Eq. (1.1) has a unique positive solution with initial condition (1.2).

When $a_n \equiv \alpha$ and $b_n \equiv \beta$ are positive constants and $F(n, x) = \beta f(x)$, Eq. (1.1) reduces to the form

$$x_{n+1} = \alpha x_n + \beta f(x_{n-k}) \tag{1.3}$$

which includes several discrete models derived from mathematical biology. The global attractivity of positive solutions of Eq. (1.3) and applications have been studied by numerous authors; see, for example, [1–5] and references cited therein.

Besides their theoretical interest, difference equations with periodic coefficients are important in mathematical biology. A model of this type could be used to mimic a population's response to seasonal fluctuations in its environment or a population with several discrete life-cycle stages. The existence of one or more periodic solutions for Eq. (1.1) and some related forms has been studied by several authors and numerous results have been obtained on this topic; see, for example, [6–9] and references cited therein. However, studies of global attractivity of periodic solutions are scarce. In this work, we obtain a sufficient condition for every positive solution of Eq. (1.1) to converge to a positive periodic solution. Applications to some discrete population models are also given.

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2. The main results

The following theorem provides a sufficient condition for every positive solution of Eq. (1.1) to converge to a positive periodic solution. For the sake of convenience, we adopt the notation $\prod_{i=m}^{n} a_i = 1$ whenever $m > n$ in the discussion.

Theorem 1. *Assume that $F(n, x)$ is nonincreasing in x, and L-Lipschitz for each $0 \le n \le p - 1$, that is, there are nonnegative constants L_n such that*

$$|F(n, x) - F(n, y)| \le L_n|x - y|, \quad n = 0, 1, \ldots, p - 1. \tag{2.1}$$

Suppose also that

$$\sum_{j=n}^{n+k} L_j \prod_{i=j+1}^{n+k} a_i < 1, \quad n = 0, 1, \ldots, p - 1. \tag{2.2}$$

Then Eq. (1.1) has a unique positive periodic solution $\{\tilde{x}_n\}$ with period p and every positive solution $\{x_n\}$ of Eq. (1.1) satisfies

$$\lim_{n\to\infty} (x_n - \tilde{x}_n) = 0. \tag{2.3}$$

Proof. Since $F(n, x) > 0$, $F(n, x)$ is periodic in n and nonincreasing in x, from the known results (see, for example, [6–9]), it is easy to see that Eq. (1.1) has a positive periodic solution $\{\tilde{x}_n\}$ with period p. Clearly, if we can show that every positive solution of Eq. (1.1) converges to $\{\tilde{x}_n\}$, then $\{\tilde{x}_n\}$ is the unique positive periodic solution. To this end, let $y_n = x_n - \tilde{x}_n$. Then $\{y_n\}$ satisfies

$$y_{n+1} = a_n y_n - (\tilde{x}_{n+1} - a_n\tilde{x}_n) + F(n, y_{n-k} + \tilde{x}_{n-k}). \tag{2.4}$$

Since $\{\tilde{x}_n\}$ is a solution of Eq. (1.1), $\tilde{x}_{n+1} - a_n\tilde{x}_n = F(n, \tilde{x}_{n-k})$. Hence, it follows that

$$y_{n+1} = a_n y_n + (F(n, y_{n-k} + \tilde{x}_{n-k}) - F(n, \tilde{x}_{n-k})). \tag{2.5}$$

First, assume that $\{x_n\}$ does not oscillate about $\{\tilde{x}_n\}$. Hence, $\{y_n\}$ is either eventually positive or eventually negative. Let us assume that $\{y_n\}$ is eventually positive. The proof for the case where $\{y_n\}$ is eventually negative is similar and will be omitted. Hence there is a positive integer n_0 such that $y_n > 0$, $n \ge n_0$. Then on noting that F is nonincreasing in x, from (2.5) we see that $y_{n+1} \le a_n y_n$, $n \ge n_0 + k$ and so it follows that

$$y_n \le \left(\prod_{i=n_0+k}^{n-1} a_i\right) y_{n_0+k}, \quad n > n_0 + k.$$

On noting that $a_n \in (0, 1]$, a_n is periodic and $a_n \not\equiv 1$, we see that $\prod_{i=n_0+k}^{n-1} a_i \to 0$ as $n \to \infty$. Hence $y_n \to 0$ as $n \to \infty$ and so (2.3) holds.

Next, assume that $\{x_n\}$ oscillates about $\{\tilde{x}_n\}$ and so $\{y_n\}$ oscillates about zero. Then there is an increasing sequence $\{n_r\}$ of positive integers such that $y_{n_1} \le 0$ and for $l = 1, 2, \ldots$,

$$y_n > 0 \quad \text{for } n_{2l-1} < n \le n_{2l}$$

and

$$y_n \le 0 \quad \text{for } n_{2l} < n \le n_{2l+1}.$$

Observe that (2.5) yields

$$\frac{y_{n+1}}{\prod_{i=0}^{n} a_i} - \frac{y_n}{\prod_{i=0}^{n-1} a_i} = \frac{1}{\prod_{i=0}^{n} a_i}(F(n, y_{n-k} + \tilde{x}_{n-k}) - F(n, \tilde{x}_{n-k})). \tag{2.6}$$

Summing from n_1 to $n - 1$ where $n_1 < n \le n_2$, we see that

$$\frac{y_n}{\prod_{i=0}^{n-1} a_i} - \frac{y_{n_1}}{\prod_{i=0}^{n_1-1} a_i} = \sum_{j=n_1}^{n-1} \frac{1}{\prod_{i=0}^{j} a_i}(F(j, y_{j-k} + \tilde{x}_{j-k}) - F(j, \tilde{x}_{j-k}))$$