Applied Mathematics Letters 26 (2013) 32–37

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A new a priori error analysis of nonconforming and mixed finite element methods

Mingxia Li [a,*], Shipeng Mao [b]

[a] School of Information Engineering, China University of Geosciences (Beijing), Beijing, 100083, PR China
[b] LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences (CAS), Beijing, 100190, PR China

ARTICLE INFO

Article history:
Received 28 September 2011
Received in revised form 6 January 2012
Accepted 3 February 2012

Keywords:
Mixed finite element
Nonconforming finite element
Error estimates
Low-regularity

ABSTRACT

In this short paper, we derive an a priori error analysis for the lowest order nonconforming and mixed finite element discretizations of the second order equation with low-regularity exact solutions only, belonging to $H^{1+s}(\Omega)$ with $s \in (0, \frac{1}{2})$. Furthermore, a robust convergence is proved even if the solution is exactly in $H^1(\Omega)$.

1. Introduction

In this note, we analyze the convergence of the lowest order nonconforming and mixed finite element approximations to low-regularity solutions of the model problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $A = (\alpha_{ij})_{i,j=1}^2$ is a given symmetric matrix function with real-value entries α_{ij}, $1 \leq i, j \leq 2$ and $f \in L^2(\Omega)$. For simplicity, we assume α_{ij} are piecewise constant functions. We assume that the matrix A is uniformly positive definite in Ω.

It is known that the above problem admits a unique solution $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ with $s > 0$, see [1]. However, in this work, we are only interested in the regularity with $0 \leqslant s < \frac{1}{2}$, which are frequently met in the interface problems or problems with nonsmooth coefficients or boundaries. In the case of conforming finite element methods approximation of (1), by Cea's Lemma [2], the error estimate in H^1 norm is bounded by the approximation errors of the finite element spaces, which can assure an $O(h^s)$ convergence rate in H^1 norm. While in the case of nonconforming finite element methods, the usual way to estimate the energy norm is by use of the second Strang's Lemma [2], where both the approximation error and the consistency error are involved. According to the classical theory developed in [3,2], the consistency error is often estimated on the edges of the elements, which results in the requirement of the regularity $u \in H^{1+s}(\Omega)$ with $s \geqslant \frac{1}{2}$. Such requirement is also needed in the error estimate of $H(\mathrm{div})$ mixed finite element methods because of the associated definition of the $H(\mathrm{div})$ interpolation operator, see [4–6].

The aim of this paper is twofold: firstly, it shows that the consistency error of the lowest order nonconforming finite element method can be controlled by its approximation error; secondly, it proves error estimates of the nonconforming and

* Corresponding author. Tel.: +86 15300329617.
E-mail addresses: limx@lsec.cc.ac.cn (M. Li), maosp@lsec.cc.ac.cn (S. Mao).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2012.02.007

mixed finite element methods under low regularity $u \in H^{1+s}(\Omega)$ with $0 < s < \frac{1}{2}$. Some contents of this paper are taken from [7]. Furthermore, the convergence is also obtained even if $s = 0$.

2. Error estimate for nonconforming finite element methods

Throughout this paper, we will adopt the standard conventions for Sobolev norms and seminorms. Let $\mathcal{J}_h$ be a regular triangulation (cf. [3,2]) of Ω, with each element K being an open triangle of size h_K, $h = \max_{K\in\mathcal{J}_h} h_K$. All the edges and vertices of the mesh are denoted by $\mathcal{E}_h$ and $\mathcal{P}_h$, respectively. $\forall E \in \mathcal{E}_h$, let $K_1, K_2 \in \mathcal{J}_h$ such that $E = K_1 \cap K_2$. We define the jumps and averages of the function v on E by

$$[v]_E = v^{K_1}|_E - v^{K_2}|_E, \qquad \{v\}_E = \frac{1}{2}(v^{K_1}|_E + v^{K_2}|_E),$$

where $v^{K_i} = v|_{K^i},\ i = 1, 2$.

Let $V = H_0^1(\Omega)$, the standard variational form of (1) is: Find $u \in V$ such that

$$(A\nabla u, \nabla v) = (f, v), \quad \forall v \in V. \tag{2}$$

Let V_h denote the associated nonconforming finite element space over $\mathcal{J}_h$. We set the discrete norm as $\|\cdot\|_h = (\sum_{K\in\mathcal{J}_h} |\cdot|_{1,K}^2)^{\frac{1}{2}}$. The well-known nonconforming P_1 triangle finite element space proposed in [8] reads as

$$V_h = \left\{ v_h \in L^2(\Omega),\, v_h|_K \in P_1(K),\ \forall K \in \mathcal{J}_h,\ \int_E [v_h]_E ds = 0,\ \forall E \in \mathcal{E}_h \right\}.$$

We consider the nonconforming finite element approximation of (2): Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h \tag{3}$$

with $a_h(u_h, v_h) = \sum_{K\in\mathcal{J}_h} \int_K (A\nabla u_h)\nabla v_h d\mathbf{x}$. The second Strang Lemma (cf. [3,2]) gives

$$\|u - u_h\|_h \leq C \left\{ \inf_{v_h\in V_h} \|u - v_h\|_h + \sup_{w_h\in V_h\setminus\{0\}} \frac{|E_h(u, w_h)|}{\|w_h\|_h} \right\}, \tag{4}$$

where $E_h(u, w_h) = a_h(u - u_h, w_h)$ denotes the consistency error.

In what follows, we shall prove that this consistency error is bounded by the approximation error, up to an arbitrary high order term! To this end, we need to introduce some quasi-interpolation operators, which can also be found in [9,7]. Given a fixed integer $k \geqslant 2$, let X_h^k be the associated k-th order Lagrange P_k finite element space, then there exists a quasi-interpolation operator $\Pi_h^k : V_h \to X_h^k$ such that for any $v_h \in V_h$,

$$\forall a \in \mathcal{P}_h, \quad \Pi_h^k v_h(a) = \begin{cases} 0, & \text{if } a \in \partial\Omega, \\ \dfrac{\sum\limits_{K\subset\omega_a} v_h(a)|_K}{N(a)}, & \text{else}, \end{cases} \tag{5}$$

$$\int_E \Pi_h^k v_h q\, ds = \int_E \{v_h\}_E q\, ds, \quad \forall q \in P_{k-2}(E),\ \forall E \in \mathcal{E}_h, \tag{6}$$

$$\int_K \Pi_h^k v_h q d\,\mathbf{x} = \int_K v_h q d\,\mathbf{x}, \quad \forall q \in P_{k-3}(K),\ \forall K \in \mathcal{J}_h, \tag{7}$$

$$\|v_h - \Pi_h^k v_h\|_{0,K}^2 + h_K^2 |\Pi_h^k v_h|_{1,K}^2 \leqslant C h_K^2 |v_h|_{1,\omega_K}^2, \quad \forall K \in \mathcal{J}_h, \tag{8}$$

where ω_a is the union of all the elements sharing a as one of their vertices with $N(a)$ as its corresponding cardinality, ω_K is the patch of elements adjacent to the triangle K.

Theorem 2.1. *Let u and u_h be the solution of (2) and (3), respectively, for any $k \geqslant 2$, we have*

$$\|u - u_h\|_h \leq C \left(\inf_{v_h\in V_h} \|u - v_h\|_h + \left(\sum_{K\in\mathcal{J}_h} h_K^2 \inf_{p\in P_{k-3}(K)} \|f - p\|_{0,K}^2 \right)^{\frac{1}{2}} \right). \tag{9}$$

Proof. By the formulation (3) and the variation formulation (2), with any $v_h \in V_h$, we can derive

$$\begin{aligned} E_h(u, w_h) &= a_h(u - u_h, w_h) = a_h(u, w_h) - (f, w_h) \\ &= a_h(u, w_h - \Pi_h^k w_h) - (f, w_h - \Pi_h^k w_h) \\ &= a_h(u - v_h, w_h - \Pi_h^k w_h) + a_h(v_h, w_h - \Pi_h^k w_h) - (f, w_h - \Pi_h^k w_h). \end{aligned} \tag{10}$$