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On the Stirling numbers associated with the meromorphic Weyl algebra

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ABSTRACT

The generalized Stirling numbers introduced recently (Mansour and Schork, 2011 [5], Mansour et al., 2011 [6]) are considered in detail for the particular case $s = 2$ corresponding to the meromorphic Weyl algebra. A combinatorial interpretation in terms of perfect matchings is given for these meromorphic Stirling numbers and the connection to Bessel functions is discussed. Furthermore, two related q-polynomial identities are derived.

1. Introduction

The Stirling numbers (of first and second kinds) are certainly among the most important combinatorial numbers as witnessed by their occurrences in many different contexts; see, e.g., [1–4] and the references contained therein. One of these interpretations is in terms of *normal ordering* special words in the *Weyl algebra* generated by the variables U, V satisfying

$$UV - VU = 1, \tag{1.1}$$

where on the right-hand side the identity is denoted by 1. A concrete representation for (1.1) is given by the operators

$$U \mapsto \frac{d}{dx}, \qquad V \mapsto X \tag{1.2}$$

acting on a suitable space of functions (where $(X \cdot f)(x) = xf(x)$). It is a classical result that the Stirling numbers of second kind $S(n, k)$ appear in the normal ordering of $(x\frac{d}{dx})^n$, or, in the variables used here, $(VU)^n = \sum_{k=1}^{n} S(n, k)V^kU^k$. Two of the present authors considered in [5] the following generalization of the relation (1.1), namely,

$$UV - VU = hV^s, \tag{1.3}$$

where $h \in \mathbb{C} \setminus \{0\}$ and $s \in \mathbb{R}$. In [6, Theorem 3.9], the following result was derived.

Proposition 1.1. *Let U, V be variables satisfying (1.3) with $s \in \mathbb{R} \setminus \{0, 1\}$ and $h \in \mathbb{C} \setminus \{0\}$. The generalized Stirling numbers $\mathfrak{S}_{s;h}(n, k)$ defined by*

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n, k)V^{s(n-k)+k}U^k \tag{1.4}$$

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have for $n \geq k \geq 1$ the explicit expression

$$\mathfrak{S}_{s;h}(n,k) = \frac{h^{n-k}s^n n!}{(1-s)^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{n + \frac{j}{s} - j - 1}{n}. \tag{1.5}$$

A thorough discussion of these numbers and their relation to other variants of Stirling numbers, in particular to those considered by Lang [7–9], can be found in [6]. Furthermore, it was shown that $\mathfrak{S}_{s;1}(n,k)$ is given for $s \in \mathbb{N}$ as an s-rook number of Goldman and Haglund [10] — as anticipated by Varvak [11]. More precisely, if we denote the s-rook numbers of the staircase board $J_{n,1}$ by $r_m^{(s)}(J_{n,1})$, then one has $\mathfrak{S}_{s;1}(n,k) = r_{n-k}^{(s)}(J_{n,1})$ [6, Theorem 5.8], which generalizes the well-known relation $S(n,k) = r_{n-k}(J_{n,1})$ for the conventional Stirling numbers of the second kind (corresponding to $s = 0$). As one of the referees pointed out, several properties of the numbers $\mathfrak{S}_{s;h}(n,k)$ stated in [6] could also have been derived from the observation that the corresponding infinite, lower triangular matrix $\mathfrak{S}_{s;h}$ is a particular *Sheffer matrix*, called the *Jabotinsky matrix* (for the latter, see [12]). Lang has derived in [7] many properties of his generalized Stirling numbers $S(r;n,k)$ (resp., $s(r;n,k)$) of the second (resp., first) kind by noting that the corresponding infinite, lower triangular matrices $\mathbf{S}(r)$ (resp., $\mathbf{s}(r)$) are examples of Jabotinsky matrices. Using this notation, the aforementioned relation between $\mathfrak{S}_{s;h}(n,k)$ and the generalized Stirling numbers of Lang can be written for $h = 1$ as $\mathfrak{S}_{s;1} = \mathbf{S}(s+1) \cdot \mathbf{s}(s)$ [5, Proposition 3.2].

In the present paper, we are concerned with the case $s = 2$, i.e., with the algebra generated by variables U, V satisfying $UV = VU + hV^2$. This is precisely the *meromorphic Weyl algebra* introduced by Diaz and Pariguan [13,14]. Due to $D(X^{-1}) = -X^{-2}$, one obtains, in analogy to (1.2), a representation [13]

$$U \mapsto -h\frac{d}{dx}, \qquad V \mapsto X^{-1}, \tag{1.6}$$

explaining the terminology "meromorphic". The parameter h enters only in a trivial fashion – see (1.5) – so we may restrict to the case $h = 1$ without loss of generality. We will denote $\mathfrak{S}_{2;1}(n,k)$ by $\mathfrak{S}(n,k)$ and call it the *meromorphic Stirling number of second kind*. Clearly, from (1.5), one finds the expression

$$\mathfrak{S}(n,k) = \frac{2^n n!}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n - j/2 - 1}{n}, \tag{1.7}$$

which may be reduced to a single term; see Theorem 2.1 below. Let us point out that (1.4) becomes, upon using (1.6) with $h = 1$,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n = \left(\frac{1}{x}\right)^{2n} \sum_{k=1}^{n} (-1)^{n-k} \mathfrak{S}(n,k) x^k \left(\frac{d}{dx}\right)^k. \tag{1.8}$$

Comparison of (1.8) with the definition of Lang's generalized Stirling numbers $S(r;n,k)$ in [7, (12)] reveals that $\mathfrak{S}(n,k) = |S(-1;n,k)|$, and the latter count unordered k-forests of ordered rooted increasing trees with vertices of any out-degree where there are n vertices altogether [9]; see also A132062 in OEIS [15]. The operator $\frac{1}{x}\frac{d}{dx}$ and its powers play an important role in the theory of Bessel functions; see [16,17]. For example, if the n-th Bessel function is denoted by $J_n(x)$, then one has for arbitrary $r \in \mathbb{N}$ the relation $x^{-n-r}J_{n+r}(x) = (-1)^r(\frac{1}{x}\frac{d}{dx})^r\{x^{-n}J_n(x)\}$; see [17, Section 17–211]. From (1.8), one immediately obtains

$$J_{n+r}(x) = \sum_{k=1}^{r} (-1)^k \mathfrak{S}(r,k) x^{n-r+k} \left\{x^{-n}J_n(x)\right\}^{(k)}.$$

Using the Pochhammer symbol $(n)_v := n(n+1)\cdots(n+v-1)$ and Leibniz's rule, one finds

$$J_{n+r}(x) = \sum_{k=1}^{r} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \mathfrak{S}(r,k)(n)_{k-l} x^{l-r} J_n^{(l)}(x).$$

Clearly, using Bessel's differential equation, one could now express $J_n^{(l)}(x)$ through $J_n'(x)$ and $J_n(x)$ in the fashion $J_n^{(l)}(x) = A_{n,l}(x)J_n'(x) + B_{n,l}(x)J_n(x)$ for some functions $A_{n,l}(x)$ and $B_{n,l}(x)$.

2. A simple expression for $\mathfrak{S}(n,k)$

If m is a positive integer, then $m!!$ will denote the product $m(m-2)\cdots 2$ if m is even and $m(m-2)\cdots 1$ if m is odd. By the convention for empty products, we will take $0!! = (-1)!! = 1$. It is possible to give an algebraic proof of the following result by using (1.7) and computing the ordinary generating function in n for the sum on the right-hand side with k fixed (see Remark 2.4 below for a shorter algebraic proof which makes use of exponential generating functions instead). Here, we provide a combinatorial proof featuring a certain subset of perfect matchings.