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A note on a nonlocal nonlinear reaction–diffusion model

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ABSTRACT

We give an application of the Crandall–Rabinowitz theorem on local bifurcation to a system of nonlinear parabolic equations with nonlocal reaction and cross-diffusion terms as well as nonlocal initial conditions. The system arises as steady-state equations of two interacting age-structured populations.

1. Introduction

In this note we consider coexistence solutions to age-structured population dynamics with diffusion, the main feature of the present work being the inclusion of nonlocal cross-diffusion terms. More precisely, we shall establish the existence of positive nontrivial solutions $u = u(a, x)$, $v = v(a, x)$ to the system

$$\partial_a u - \mathrm{div}_x\big(d_1(\hat{V})\nabla_x u + u\nabla_x d_2(\hat{V})\big) = -\alpha u^2 - \mu_1(\hat{V})u, \quad a \in (0, a_m), x \in \Omega, \tag{1.1}$$

$$\partial_a v - \mathrm{div}_x\big(d_3(\hat{U})\nabla_x v + v\nabla_x d_4(\hat{U})\big) = -\beta v^2 - \mu_2(\hat{U})v, \quad a \in (0, a_m), x \in \Omega, \tag{1.2}$$

for $a \in (0, a_m)$, and $x \in \Omega$, subject to the nonlocal initial conditions

$$u(0, x) = \eta U, \quad x \in \Omega, \tag{1.3}$$

$$v(0, x) = \xi V, \quad x \in \Omega, \tag{1.4}$$

and Dirichlet boundary conditions

$$u(a, x) = 0, \quad a \in (0, a_m), x \in \partial\Omega, \tag{1.5}$$

$$v(a, x) = 0, \quad a \in (0, a_m), x \in \partial\Omega, \tag{1.6}$$

where we agree here and in the following upon the notation

$$\hat{U} := \int_0^{a_m} \omega(a)u(a, \cdot)\,da, \qquad U := \int_0^{a_m} b(a)u(a, \cdot)\,da \tag{1.7}$$

for the function u defined on $J := [0, a_m]$ and analogously for the function v. Eqs. (1.1)–(1.7) arise naturally as steady-state (i.e. time-independent) equations of two age-structured populations with densities u and v, respectively, and maximal age $a_m > 0$ living in a (bounded and smooth) domain $\Omega \subset \mathbb{R}^n$, where a is the age and x is the space variable. The integrals with respect to age in (1.1) and (1.2) are (weighted) local total populations with a given nonnegative weight function ω.

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The divergence terms in (1.1) and (1.2) describe spatial movement with nonlocal coefficients d_j. They reflect intrinsic dispersion as well as an increase of dispersive forces by repulsive or attractive interferences with an increase of the other population. We refer to [1] for a derivation of such kind of models (without age-structure). The right hand sides of (1.1) and (1.2) take into account intra- and inter-specific interactions of the two populations with constants $\alpha, \beta > 0$ and functions μ_j depending nonlocally on the population densities. Creation of new individuals is described by (1.3) and (1.4) with birth profile b and parameters η, ξ measuring the intensity of the fertility. We refer to [2,3] for further information on the modeling assumptions. To avoid unnecessary notational complications, the equations above are stated as a simplified version of more elaborate models, and we remark that the subsequent analysis would not change in any way if one would allow for e.g. different weight functions, different birth rates, or different maximal ages for the two populations.

In this note we give an extension of previous results [3–5] being described in more detail in the next section. Steady-states for a single age-structured populations were investigated e.g. in [6]. We shall also point out that steady-state solutions for two interacting populations when age-structure is neglected, i.e. variants of the elliptic counterparts of (1.1)–(1.2), have attracted considerable interest in the past, see for example [7–13] and the references therein.

2. Notation and main result

The main features of the equations under consideration are the nonlocalities appearing in the diffusion and reaction terms as well as in the initial conditions. Similar equations with local reaction terms (i.e. $\mu_1(v) = \alpha_2 v$, $\mu_2(u) = \pm\beta_2 u$ with $\alpha_2, \beta_2 > 0$) have been investigated in [3,4] with linear diffusion (i.e. $d_1 = d_3 = 1$ and $d_2 = d_4 = 0$) and in [5] with a local cross-diffusion term (i.e. $d_1(v) = 1 + v$, $d_2(v) = v$, $d_3 = 1$, and $d_4 = 0$). In these papers global bifurcation results have been derived with respect to the parameters η and ξ. The aim of this note is to show that (local) bifurcation results can be obtained for Eqs. (1.1)–(1.7) including nonlinear nonlocal diffusion terms. More precisely, we shall provide values for the parameters η and ξ for which (1.1)–(1.7) have *coexistence solutions*, i.e. smooth solutions (u, v) with both components nontrivial and positive. Establishing positive steady-state solutions is a first step toward an understanding of (time-dependent) two population dynamics. To this end, we assume throughout the paper that

$$d_1, d_3, \mu_1, \mu_2 \in C^1(\mathbb{R}), \qquad d_2, d_4 \in C^2(\mathbb{R}), \tag{2.1}$$

satisfy

$$d_2(0) = \mu_1(0) = 0, \tag{2.2}$$

respectively,

$$d_j(z) \geq \delta, \quad z \geq 0, j = 1, 3, \tag{2.3}$$

for some $\delta > 0$. For an easier reference in the future we suppose that

$$d_1(0) = 1. \tag{2.4}$$

For the weight and the birth functions we assume

$$\omega, b \in L_\infty^+(J), \quad b(a) > 0 \text{ for } a \text{ near } a_m \tag{2.5}$$

together with the normalization

$$\int_0^{a_m} b(a) e^{-\lambda_1 a}\, da = 1, \tag{2.6}$$

where $\lambda_1 > 0$ denotes the principal eigenvalue of $-\Delta_x$ on Ω subject to Dirichlet boundary conditions. For technical reasons we introduce $\mathbb{L}_q := L_q(J, L_q(\Omega))$ and the solution space

$$\mathbb{W}_q := L_q(J, W^2_{q,D}) \cap W^1_q(J, L_q)$$

with $q \in (n+2, \infty)$ fixed, where $L_q := L_q(\Omega)$, and $W^\kappa_{q,D} := W^\kappa_{q,D}(\Omega)$ refer to Sobolev–Slobodeckij spaces including Dirichlet boundary conditions if meaningful, i.e. if $\kappa > 1/q$. We let $\mathbb{W}^+_q$ denote the positive cone of $\mathbb{W}_q$ and set $\dot{\mathbb{W}}^+_q := \mathbb{W}^+_q \setminus \{0\}$. Recall the embedding

$$\mathbb{W}_q \hookrightarrow C^{1-1/q-\vartheta}\big([0, a_m], W^{2\vartheta}_{q,D}\big), \quad 0 \leq \vartheta \leq 1 - 1/q, \tag{2.7}$$

which holds for $\vartheta = 1 - 1/q$ due to [14, III. Theorem. 4.10.2] and otherwise by the interpolation inequality [14, I. Theorem. 2.11.1]. In particular, the trace $\gamma_0 u := u(0)$ defines a bounded linear operator $\gamma_0 \in \mathcal{L}(\mathbb{W}_q, W^{2-2/q}_{q,D})$. Also recall (e.g. from [14]) that $A \in L_\infty(J, \mathcal{L}(W^2_{q,D}, L_q))$ is said to have *maximal L_q-regularity* provided that the operator

$$(\partial_a + A, \gamma_0) \in \mathcal{L}(\mathbb{W}_q, \mathbb{L}_q \times W^{2-2/q}_{q,D})$$