Applied Mathematics Letters 25 (2012) 1778–1783

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Compactness and essential norms of composition operators on Orlicz–Lorentz spaces

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ARTICLE INFO

Article history:
Received 25 January 2011
Accepted 7 February 2012

Keywords:
Essential norm
Compact operators
Composition operators
Orlicz–Lorentz spaces

ABSTRACT

In this work, we present necessary and sufficient conditions for compactness of the composition operator on Orlicz–Lorentz spaces and determine upper and lower estimates for the essential norm of the composition operator on these spaces.

1. Introduction

An Orlicz function $\varphi : [0,\infty) \mapsto [0,\infty)$ is a convex function with $\varphi(u) = 0$ iff $u = 0$, $\varphi(u) \to \infty$ as $u \to \infty$ and $\lim_{u\to b(\varphi)_-} \varphi(u) = \varphi(b(\varphi))$, where $b(\varphi) = \sup\{u > 0 : \varphi(u) < \infty\}$; see [1–5]. Let (Ω, Σ, μ) be a σ-finite and complete measure space and let $L^\circ(\mu)$ denote the linear space of all equivalence classes of Σ-measurable functions on Ω that are identified μ-a.e. Let $M_\circ$ be the class of all functions in $L^\circ(\mu)$ that are finite μ a.e.

For $f \in M_\circ$, the distribution function μ_f of f on $(0,\infty)$ is defined as (see [6,2])

$$\mu_f(\lambda) = \mu\{x \in \Omega : |f(x)| > \lambda\},$$

and the decreasing rearrangement of f on $(0,\infty)$ is defined as

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\} = \sup\{\lambda > 0 : \mu_f(\lambda) > t\}.$$

Let $I = [0, a]$ if $a < \infty$ and $I = [0, a)$ if $a = \infty$, where $a = \mu(\Omega)$. Let $w : I \mapsto (0,\infty)$ be a weight function which is non-increasing and locally integrable on I with respect to the Lebesgue measure.

The Orlicz–Lorentz space $\Lambda^{\varphi,w}(\mu)$ is defined as

$$\Lambda^{\varphi,w}(\mu) = \{f \in L^\circ(\mu) : \rho_{\varphi,w}(\lambda f) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_{\varphi,w}(\lambda f) = \int_I \varphi(\lambda f^*(t)) w(t) dt.$$

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0893-9659/$ – see front matter
doi:10.1016/j.aml.2012.02.011

The Orlicz–Lorentz space $\Lambda^{\varphi,w}(\mu)$ is a Banach function space under the norm

$$\|f\|_{\varphi w} = \inf\{\lambda > 0 : \rho_{\varphi,w}(f/\lambda) \le 1\};$$

see [7,8,1,2].

If $w \equiv 1$, then $\Lambda^{\varphi,w}(\mu)$ is the Orlicz space $L^{\varphi}(\mu)$ and if $\varphi(t) = t$, then $\Lambda^{\varphi w}(\mu)$ is the Lorentz space $\Lambda^{w}(\mu)$.

The Lorentz spaces $L^{pq}(\mu)$ are defined as

$$L^{pq}(\mu) = \{f \in L(\mu) \colon \|f\|_{L^{pq}} < \infty\},$$

where

$$\|f\|_{L^{pq}} = \begin{cases} \left[\int_0^{\infty} (t^{1/p} f^*(t))^q dt/t\right]^{1/q} & \text{if } 1 < p < \infty,\ 1 \le q < \infty, \\ \sup\limits_{0<t<\infty} \ t^{1/p} f^*(t) & \text{if } 1 < p < \infty,\ q = \infty. \end{cases}$$

In the case where $1 < p < \infty$ and $1 \le q < \infty$, the space $L^{pq}(\mu)$ is equal to $\Lambda^{\varphi,w}(\mu)$ with $\varphi(u) = |u|^q$ for all $u \in \mathbb{R}$ and $w(t) = t^{(\frac{1}{p}-1)\frac{1}{q}}$ for all $t \in [0, \infty)$.

We recall that $\varphi \in \Delta_2(\mathbb{R})$ if there exists $k > 0$ such that $\varphi(2x) \le k\varphi(x)$ for $x \ge 0$. We say that $\varphi \in \Delta_2(\infty)$ if there exists $u_0 > 0$ with $\varphi(u_0) < \infty$ and $k > 0$ such that $\varphi(2u) \le k\varphi(u)$ whenever $|u| \ge u_0$. We say that $\varphi \in \Delta_2$ whenever $\varphi \in \Delta_2(\mathbb{R})$ if μ is nonatomic and infinite and $\varphi \in \Delta_2(\infty)$ if μ is nonatomic and finite. If φ^*, the Young conjugate of φ, satisfies condition $\Delta_2(\mathbb{R})$ or $\Delta_2(\infty)$, we write $\varphi \in \nabla_2(\mathbb{R})$ or $\varphi \in \nabla_2(\infty)$, respectively.

For details about Orlicz–Lorentz spaces we refer the reader to [6,9,7,8,1,12].

A mapping $T : \Omega \mapsto \Omega$ is said to be measurable if $T^{-1}(A) \in \Sigma$ whenever $A \in \Sigma$. A measurable transformation $T : \Omega \mapsto \Omega$ is called non-singular if the preimage of every null set under T is a null set. Such a transformation induces a well-defined composition operator

$$C_T : L^{\circ}(\mu) \mapsto L^{\circ}(\mu)$$

defined by

$$C_T f = f \circ T, \quad \text{for each } f \in L^{\circ}(\mu).$$

In the case where C_T maps $\Lambda^{\varphi,w}(\mu)$ into itself, we call C_T a composition operator on $\Lambda^{\varphi,w}(\mu)$ induced by T.

The study of composition operators on Lorentz spaces and Orlicz spaces was initiated in [10,11], and [5, p. 368]. For composition operators on Orlicz–Lorentz spaces, see [12]. For a study of composition operators on other spaces, see [13,10,14,11,12,15–18] and the references therein.

Recall that the essential norm of an operator T is defined as

$$\|T\|_e = \inf\{\|T - K\| : K \text{ being a compact operator}\}.$$

We know that $\|T\|_e = 0$ if and only if T is compact.

In this work we study the essential norm of composition operators on Orlicz–Lorentz spaces.

2. The essential norm

The following theorem gives a necessary and sufficient condition for the compactness of the composition operators on Orlicz–Lorentz spaces.

Theorem 2.1. *Let φ be an Orlicz function. Then $C_T : \Lambda^{\varphi,w}(\mu) \to \Lambda^{\varphi,w}(\mu)$ is a compact operator if and only if for each $\varepsilon > 0$ the set*

$$N = N(h, \varepsilon) = \{x \in \Omega : h(x) > \varepsilon\}$$

consists of only finitely many atoms, where $h(x) = d\mu \circ T^{-1}(x)/d\mu(x)$.

Proof (*Necessity*). Suppose there exists some $\varepsilon > 0$ such that $N(h, \varepsilon)$ either contains a non-atomic measurable subset or has countably many atoms. Since $N(h, \varepsilon) \subset N(h, \delta)$, if $0 < \delta < \varepsilon$, we can assume that $0 < \varepsilon \le 1$. In both cases, we can find a sequence $\{A_n\}$ of pairwise disjoint measurable subsets of $N(h, \varepsilon)$ with $0 < \mu(A_n) < \infty$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, define

$$f_n(x) = \frac{\chi_{A_n}(x)}{\|\chi_{A_n}\|_{\varphi,w}}.$$