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Global existence and optimal decay rate for the strong solutions in H^2 to the compressible Navier–Stokes equations ☆

Yanjin Wang *, Zhong Tan

School of Mathematical Sciences, Xiamen University, Fujian 361005, China

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ABSTRACT

We prove the global existence of a unique strong solution to the compressible Navier–Stokes equations when the initial perturbation is small in H^2. If further that the L^1 norm of initial perturbation is finite, we prove the optimal L^2 decay rates for such a solution and its first-order spatial derivatives.

1. Introduction

In this paper we consider the compressible Navier–Stokes equations for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u - \lambda\nabla \operatorname{div} u = 0, \tag{1.2}$$

which governs the motion of a compressible viscous baratropic fluid, where ρ, u represent the density, velocity of the fluid respectively and $p = p(\rho)$ is the pressure satisfying $p'(\rho) > 0$ for $\rho > 0$. The constants $\mu > 0,\ \lambda \geq 0$ are the viscosity coefficients. We supplement the system by the initial data

$$\rho|_{t=0} = \rho_0, \qquad u|_{t=0} = u_0. \tag{1.3}$$

There are a lot of mathematical results about the existence and large-time behavior of the solutions to the compressible Navier–Stokes equations. Among them, when there is no vacuum initially, Matsumura and Nishida [1,2] proved the first global existence of solutions in H^3 (classical solutions) for the small initial data and local existence for the general large initial data, both in the whole space $\mathbb{R}^3$ and in a general domain with boundary. Later, Valli [3] and Kawashita [4] obtained the similar existence results of strong solutions in H^2. The time-decay rate of the solutions obtained in [1] was investigated in [5–7] and the references therein for instance. In particular, it was proved in [7] that the optimal $L^p,\ 2 \leq p \leq 6$ decay rate of the solutions and the optimal L^2 decay rates of the first-order derivatives. However, all the previous decay rates were proved for the solutions in H^3 or more regular solutions. In this paper, we will refine the works [1,3,7] to show the global existence and optimal decay rates for the strong solutions to the Cauchy problem (1.1)–(1.3) in the H^2 framework. Main results of this paper are stated as the following theorem.

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* Corresponding author.
E-mail address: wangyanjin_2008@163.com (Y. Wang).

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Theorem 1.1. *Let $\bar{\rho}$ be a positive constant, there exists a constant δ_0 such that if*

$$|(\rho_0-\bar{\rho},u_0)|_{H^2}\le\delta_0, \tag{1.4}$$

then there exists a unique global solution (ρ,u) of the Cauchy problem (1.1)–(1.3) satisfying

$$|(\rho-\bar{\rho},u)(t)|_{H^2}^2+\int_0^t|\nabla\rho(s)|_{H^1}^2+|\nabla u(s)|_{H^2}^2\mathrm{d}s\le C|(\rho_0-\bar{\rho},u_0)|_{H^2}^2,\quad t\ge 0. \tag{1.5}$$

If further, $(\rho_0-\bar{\rho},\ u_0)\in L^1$, then

$$|\nabla(\rho-\bar{\rho},u)(t)|_{H^1}\le C_0(1+t)^{-\frac{5}{4}}, \tag{1.6}$$

$$|(\rho-\bar{\rho},u)(t)|_{L^p}\le C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})},\quad \forall p\in[2,6]. \tag{1.7}$$

The key assumption in Theorem 1.1 and also in [1–3] is that the density is bounded away from the vacuum, see also [8,9] the existence of weak solutions for small discontinuous initial data. When the vacuum is allowed, the breakthrough global existence of finite energy weak solutions was proved by Lions [10] for $p=\rho^\gamma$ with $\gamma\ge\frac{9}{5}$ and was later generalized by Feireisl [11] for $\gamma>\frac{3}{2}$. Salvi and Straškraba [12], Choe and Kim [13] proved the local existence of strong solutions provided that the initial data satisfy a natural compatibility condition, and recently, Cho and Kim [14] proved the local existence of classical solutions. However, one cannot expect in general the global existence of H^2 or H^3 solutions due to the blow-up results of Xin [15], Cho and Jin [16]. Only very recently, the first global classical solutions which may have large oscillations and can contain vacuum states were obtained in [17] under the small initial energy assumption.

The rest of this paper is devoted to prove Theorem 1.1. We briefly explain the strategy of the proof. (i) For the existence part, it suffices to derive (1.5) a priori, then the global existence will follow in a standard way as in [1] by the local existence, a priori estimates and the continuity argument. The proof of (1.5) is based on the revision of the papers [1–3]: first, we bound $|\nabla^2u|_{H^1}$ in terms of $\delta|\nabla\rho|_{H^1}$ by balancing the linear term $\frac{p'(\bar{\rho})}{\bar{\rho}}\nabla\rho$ and $\bar{\rho}\mathrm{div}u$ with each other as in [2,3], but we do not estimate for ρ_t, u_t and hence the estimates will be much simpler and clearer; second, we directly use the equations to bound $|\nabla\rho|_{H^1}$ in terms of $|\nabla u|_{H^2}$; then finally, combining them with the elementary energy identity, we close the estimates and obtain (1.5). (ii) For the decay rates, as in [7], we first derive a Lyapunov-type energy inequality which involves $|(\nabla\rho,\nabla u)|_{L^2}$. Then, we apply the linear decay estimates to estimate the time-decay rate of $|(\nabla\rho,\nabla u)|_{L^2}$ in terms of the energy functional, hence the estimate is closed and (1.6)–(1.7) follows. However, we shall make full use of the inequality (1.5) to modify the proof in [7] due to the fact that $|\nabla^3u|_{L^2}$ is not included in the energy functional.

In this paper, we use the standard notations L^p, H^s to denote the L^p and Sobolev spaces on $\mathbb{R}^3$, with norms $|\cdot|_{L^p}$ and $|\cdot|_{H^s}$, respectively. We use C to denote the constants depending only on the physical coefficients and denote C_0 to be constants depending additionally on the initial data.

2. Proof of Theorem 1.1

We prove our theorem in this section. Since the local strong solutions can be proven by the standard argument of the contracting map theorem as [1,3,4] whose details we omit, global solutions will follow in a standard continuity argument after we establish (1.5) a priori. Therefore, we assume a priori that

$$|(\rho-\bar{\rho},u)(t)|_{H^2}\le\delta\ll 1. \tag{2.1}$$

This together with Sobolev's inequality implies in particular that

$$\bar{\rho}/2\le\rho\le 2\bar{\rho}. \tag{2.2}$$

This should be kept in mind in the rest of this paper.

The first estimate is the elementary energy identity whose proof we omit for simplify

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|u|^2+2G(\rho)\mathrm{d}x+\int_{\mathbb{R}^3}\mu|\nabla u|^2+\lambda|\mathrm{div}u|^2\mathrm{d}x=0, \tag{2.3}$$

where the relative potential energy $G(\rho)$ is defined by

$$G(\rho)=\rho\int_{\bar{\rho}}^{\rho}\frac{p(s)-p(\bar{\rho})}{s^2}\mathrm{d}s.$$

Then we have $G'(\bar{\rho})=0,\ G''(\bar{\rho})>0$ and hence in the neighborhood of $\bar{\rho}$, we have

$$C^{-1}|\rho-\bar{\rho}|^2\le G(\rho)\le C|\rho-\bar{\rho}|^2. \tag{2.4}$$