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Adaptive fourth-order partial differential equation filter for image denoising

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ABSTRACT

To overcome the staircasing effects and simultaneously avoid edge blurring, this paper describes a fourth-order partial differential equation based edge-preserving regularization filter for noise removal. This technique is closely related to the nonlinear anisotropic diffusion. Compared results distinctly demonstrate the superiority of our proposed scheme over the LLT model, in terms of removing noise while sharply maintaining the edge features.

1. Introduction

Removing noise while preserving fine details is a challenging issue in image processing. One classical partial differential equation (PDE) based technique is the total variation (TV) minimization, which was inaugurated in [1] by Rudin et al. depicted as

$$\min_{u\in BV(\Omega)\bigcap L^2(\Omega)} \int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 \mathrm{d}x, \tag{1.1}$$

where Ω denotes a bounded open domain with a Lipschitzian boundary, u and u_0 represent the original image and the observed image respectively. Furthermore, to improve the edge-preserving capability, Strong and Chan [2,3] presented the adaptive TV approach to image restoration

$$\min_{u\in BV(\Omega)\bigcap L^2(\Omega)} \int_\Omega \alpha(x)|Du| + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 \mathrm{d}x \tag{1.2}$$

where $\alpha(x)$ is a spatially and scale adaptive function. In [4], Chen and Wunderli chose $\alpha(x) = \frac{1}{1+\mathcal{K}|\nabla G_\sigma * u_0|^2}$ as an edge-stopping function, used for controlling the speed of the diffusion, where $\mathcal{K}$ represents a threshold parameter, and $G_\sigma(x) = \frac{1}{2\pi\sigma^2}\exp(-\frac{|x|^2}{2\sigma^2})$ denotes the Gaussian filter with parameter σ. Additionally, the corresponding theory of viscosity solutions was investigated there in detail.

These schemes are capable of suppressing noise while preserving the sharp edges. Unfortunately, the numerous blocky effects are engendered in the recovered images owing to the TV regularization. To overcome this drawback, high-order

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PDE filter techniques have been researched by scholars widely. Ref. [5] exhibits the superiority of fourth-order PDE over second-order PDE in image processing. Recently, great success has been achieved, for instance, fourth-order anisotropic diffusion strategy [5–10], and fourth-order PDE regularization minimization scheme [11–15], etc. Thereinto, the classical LLT model [12] can be characterized by the following formulation

$$\min_{u\in BV^2(\Omega)\bigcap L^2(\Omega)} \int_\Omega |D^2u| + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 \mathrm{d}x. \tag{1.3}$$

Numerical experiments demonstrate that the LLT model can relieve the staircasing effects substantially. A fly in the ointment is that this fourth-order filter results frequently in edge blur. In order to conquer the staircase artifacts and simultaneously avoid the blurring effects, the hybrid regularization approaches combining TV filter and fourth-order PDE filter were displayed in [16,17].

Motivated by the above models (1.2) and (1.3), we primitively introduce the adaptive fourth-order PDE regularization based image restoration scheme

$$\min_{u\in BV^2(\Omega)\bigcap L^2(\Omega)} \int_\Omega \alpha(x)|D^2u| + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 \mathrm{d}x \tag{1.4}$$

where $\alpha(x) = \frac{1}{1+\mathcal{K}|\nabla G_\sigma * u_0|^2}$ denotes the diffusivity function, which adaptively manipulates the amount of smoothing. More precisely, for given $\mathcal{K}$, in homogeneous flat regions where $|\nabla G_\sigma * u_0|$ is small, which indicates a strong diffusion process. Conversely, near the region's boundaries where $|\nabla G_\sigma * u_0|$ is large, the diffusion process is weak. In summary, the diffusivity function can effectively prevent the diffusion of the edges and sharply preserve the edges while removing noise.

Our principal contributions of this research are to address a modified fourth-order PDE filtering scheme, and improve the quality of the recovered images substantially, in term of overcoming the staircasing effects and preserving the sharp edges. The remainder of this paper is arranged as follows. In Section 2, we describe the necessary definitions and notions about the model (1.4). Section 3 elaborates on the numerical method for our novel strategy. And numerical experiments intended for demonstrating the proposed method are provided in Section 4. Finally, conclusions are summarized in Section 5.

2. Preliminaries

Our objective in this section is to describe some necessary definitions and notions for the proposed model. Following the Refs. [17,18], we start with the precise definition of BV^2 space.

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open domain with Lipschitz boundary. Let $u \in L^1(\Omega)$. Then the BV^2 semi-norm of u is characterized by

$$\int_\Omega |D^2u| = \sup_{\phi\in C_c^2(\Omega,\mathbb{R}^{n\times n})} \left\{ \int_\Omega \sum_{i,j=1}^n u\partial_j\partial_i\phi^{ij}\mathrm{d}x : |\phi(x)| \le 1,\ \forall x\in\Omega \right\} < \infty, \tag{2.1}$$

where $\phi(x)$ is a vector valued function, with $|\phi(x)| = \sqrt{\sum_{i,j=1}^n (\phi^{ij})^2}$. Here we remark that the space $BV^2(\Omega)$ equipped with $\|u\|_{BV^2(\Omega)} = \int_\Omega |D^2u| + \|u\|_{L^1(\Omega)}$ is also a Banach space.

Definition 2.2. Suppose that $\Omega \subseteq \mathbb{R}^n$ be a bounded open domain with Lipschitz boundary, and $u \in L^1(\Omega)$. Suppose also that $\alpha(x) \ge 0$ represents a continuous and real function. Then we define the weighted BV^2 semi-norm of u as

$$\int_\Omega \alpha|D^2u| = \sup_{\phi\in C_c^2(\Omega,\mathbb{R}^{n\times n})} \left\{ \int_\Omega \sum_{i,j=1}^n u\partial_j\partial_i\phi^{ij}\mathrm{d}x : |\phi(x)| \le \alpha,\ \forall x\in\Omega \right\} < \infty, \tag{2.2}$$

and the α-BV^2 norm to be $\|u\|_{\alpha-BV^2(\Omega)} = \int_\Omega \alpha|D^2u| + \|u\|_{L^1(\Omega)}$.

Similarly to the Ref. [4], we can derive the following propositions.

Proposition 2.1 (*Lower Semicontinuity*). *Assume $\{u_i\}_{i=1}^\infty \subset BV^2(\Omega)$ and $u_i \to u^*$ in $L^1(\Omega)$, then*

$$\int_\Omega \alpha|D^2u^*| \le \liminf_{i\to\infty}\int_\Omega \alpha|D^2u_i|. \tag{2.3}$$