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W solutions of the CW equation for flow friction

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ABSTRACT

The empirical Colebrook–White (CW) equation belongs to the group of transcendental functions. The CW function is used for the determination of hydraulic resistances associated with fluid flow through pipes, flow of rivers, etc. Since the CW equation is implicit in fluid flow friction factor, it has to be approximately solved using iterative procedure or using some of the approximate explicit formulas developed by many authors. Alternate mathematical equivalents to the original expression of the CW equation, but now in the explicit form developed using the Lambert W-function, are shown (with related solutions). The W-function is also transcendental, but it is used more general compared with the CW function. Hence, the solution to the W-function developed by mathematicians can be used effectively for the CW function which is of interest only for hydraulics.

1. Introduction

The problem of flow in pipes and open conduits was one which had been of considerable interest to engineers for nearly 250 years. Even today, this problem is not solved definitively [1]. The difficulty to solve the turbulent flow problems lies in the fact that the friction factor is a complex function of relative surface roughness (ε/D) and the Reynolds number (Re). Precisely, hydraulic resistance depends on flow rate. Similar situation is with electrical resistance when diode is in circuit. To be more complex, widely used, empirical Colebrook–White (CW) equation valuable for the determination of hydraulic resistances for turbulent regime in smooth and rough pipes is iterative (implicit in fluid flow friction factor). The unknown friction factor appears on both sides of the equation, i.e., both the right- and left-hand terms contain friction factor [2,3]. The Colebrook–White equation is also known as the Colebrook equation or simply the CW equation (1):

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10}\left(\frac{2.51}{\mathrm{Re} \cdot \sqrt{\lambda}} + \frac{\varepsilon}{3.71 \cdot D}\right). \tag{1}$$

The empirical implicit CW equation itself can produce an error of more than 5% but today even apropos this fact, it is the accepted standard for the calculation of flow friction factor in hydraulically smooth and rough pipes [4]. Many researchers adopt a modification of the CW equation, using the 2.825 constant instead of 2.51 especially for gas flow calculation [5,6]. This adoption for gas flow produced a deviation of maximal 3.2% compared with the classical CW equation.

As an alternative to the implicit CW equation, many approximate explicit formulas were given. Gregory and Fogarasi [7], Yıldırım [8] and Brkić [9] made comparisons of the available approximations of the CW equation at that time.

2. Explicit reformulation of the CW equation based on the Lambert W-function

Lambert W and CW are transcendental functions. The (real-valued) Lambert W-function is a solution of the nonlinear equation $W \cdot e^W = x$. The range of the lower branch of the inverse Lambert function is $-1 \leq W_{-1}$, while the upper branch

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Notations

λ	Darcy friction factor (dimensionless)
Re	Reynolds number (dimensionless)
D	Inner pipe diameter (m)
ε	Roughness of the inner surface of the pipe (m)
ε/D	Relative roughness of the inner surface of the pipe (dimensionless)
W	Lambert function
ω	Shifted, auxiliary function proposed by Boyd
x	Argument of the Lambert W-function
y	Argument of the shifted, auxiliary function proposed by Boyd
n	Positive integer number
ϖ, Ω, ζ, S	Auxiliary terms defined in text.

W_0 is divided into $-1 \le W_0^- \le 0$ and $0 \le W_0^+$. W_0 is referred to as the principal branch of the Lambert W-function. Only W_0^+ part of the principal branch of the Lambert W-function will be used for the solution of the problem presented here. The Lambert function is implemented in many mathematical systems like Mathematica by Wolfram Research under the name ProductLog or Matlab by MathWorks under the name Lambert. Note that the name "W" for the Lambert function is not as old as the related function [10]. The modern history of Lambert W began in the 1980s, when a version of the function was built into the Maple computer-algebra system and given the name W. Corless et al. [11] proposed the name Lambert W for this function and this name is also used here. But in formulas only the letter W is used for related function because this notation is shorter. The Lambert W-function is somewhere known as Omega [11]. For the CW equation, only the positive part of the principal branch of the Lambert W-function is considered (Fig. 1) because the other branches correspond to nonphysical solutions of CW equation, so the simplified notation W is not ambiguous. Fortuitously, the letter W has additional significance because of the pioneering work on many aspects of W by Wright [12]. Although White was not actually a co-author of the paper in which CW equation was presented [2], Colebrook made a special point of acknowledging important contribution of White to the development of the equation [3]. So letter W has additional symbolic value in the CW equation (2) reformulated here:

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10}\left(10^{\frac{-W(x)}{\ln(10)}} + \frac{\varepsilon}{3.71 \cdot D}\right) = -2 \cdot \log_{10}\left(\frac{5.02 \cdot W(x)}{\mathrm{Re} \cdot \ln(10)} + \frac{\varepsilon}{3.71 \cdot D}\right). \tag{2}$$

Here, argument of the Lambert W-function can be noted as (3)

$$x = \frac{\mathrm{Re} \cdot \ln(10)}{5.02}. \tag{3}$$

In the papers of Moore [13], Nandakumar [14] and Goudar and Sonnad [15–17], other possible transformations of the CW equation using the Lambert-W function are shown. But relations shown in these papers have limitation in applicability for high values of Reynolds number and relative roughness because the computers available today cannot operate with extremely large numbers [18]. In the paper of Keady [19], the CW equation is expressed in Maple notation using the Lambert W-function. Clamond [20] also provides Matlab and FORTRAN codes for CW relation expressed in terms of the Lambert W-function.

3. Possible solutions of the CW equation based on the Lambert W-function

Besides the relative simplicity of an explicit form of the CW equation transformed using the Lambert W-function, it allows highly accurate estimation of friction factor as the Lambert W-function can be evaluated accurately [21].

3.1. Formal solution

Since the Lambert W is a transcendental function, formal solution of the Lambert W-function can be expressed only in endless form (4):

$$W(x) = \ln \frac{x}{\ln\left(\frac{x}{\ln\left(\frac{x}{\ln(\ddots)}\right)}\right)}. \tag{4}$$