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Common fixed points of (ψ, ϕ)-type contractive maps

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ABSTRACT

One interesting technique for obtaining fixed point results is the technique of contractive conditions of integral type. (ψ, ϕ)-type contractive maps are introduced in order to generalize this technique. Some common fixed point results for (ψ, ϕ)-type contractive maps on metric spaces are proved. Finally, a result is also obtained concerning the discontinuity of (ψ, ϕ)-type contractive maps at their unique common fixed point.

1. Introduction

The study of common fixed points of compatible mappings has been an active area of research interest ever since Jungck [1] introduced the notion of compatible mappings in 1986. Let f and g be self-maps of a metric space $X := (X, d)$ and $C(f, g) := \{x \in X : fx = gx\}$. The maps f and g are called compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}_{n\geq 1}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. We say that the maps f and g have the property $(E.A)$ [2] if there exists a sequence $\{x_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. If two maps are noncompatible, then they satisfy the property $(E.A)$, but the converse is not necessarily true [2,3]. Pant [4] introduced the notion of pointwise R-weakly commuting mappings and he proved, in [5], that pointwise R-weak commutativity is equivalent to commutativity at coincidence points. Jungck [6] defined f and g to be weakly compatible if $fgx = gfx$ for all $x \in C(f, g)$. Note that f and g are weakly compatible if and only if f and g are pointwise R-weakly commuting.

One interesting technique for obtaining fixed point results is the technique of contractive conditions of integral type. In [7], trying to extend a theorem of Branciari [8], Rhoades established two fixed point theorems satisfying a contractive inequality of integral type. Also, Djoudi et al. used the concept of weak compatibility for obtaining the common fixed point of mappings satisfying a contractive condition of integral type and the maps are not necessary continuous [9,10]. In this work, we generalize this useful technique by introducing (ψ, ϕ)-type contractive maps and prove some common fixed point results for such maps in metric spaces.

Let (X, d) be a metric space, $x_0 \in X$, f and g self-maps on X, $g(X) \subseteq f(X)$ and $y_n = f(x_{n+1}) = g(x_n)$ for all $n \geq 0$. Define the sets $O(y_k, n) := \{y_k, y_{k+1}, \ldots, y_{k+n}\}$ $(k \geq 0)$ and $O(y_k, \infty) := \{y_0, y_1, \ldots, y_n, \ldots\}$. The set $O(y_k, n)$ is called the nth orbit of y_k. For any set A, $\delta(A)$ will denote the diameter of A. Finally, we put

$$M(x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$$

and

$$N(x) = \max\{d(gx, gfx), d(g^2x, gfx), d(fx, gx), d(f^2x, gfx), d(fx, gfx), d(gx, f^2x)\},$$

for all $x, y \in X$.

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2. The main results

Let (X, d) be a metric space, f and g self-maps on X, $\psi : [0, \infty) \to [0, \infty)$ a nondecreasing continuous function such that $\psi(t) \geq t$ whenever $t > 0$, and $\phi : [0, \infty) \to [0, \infty)$ a function that is continuous from the right and nondecreasing for which $\phi(t) < t$ for all $t > 0$. Then, we say that f and g satisfy the (ψ, ϕ)-type contractive condition whenever

$$\psi(d(gx, gy)) \leq \phi(\psi(M(x, y)))$$

for all $x, y \in X$. Also, we say that f and g satisfy the (f, g, ϕ)-type contractive condition whenever $d(fx, fy) \leq \phi(d(gx, gy))$ for all $x, y \in X$. It is known that if $t > 0$, $\phi(t) < t$ if and only if $\lim_{n\to\infty} \phi^n(t) = 0$, where ϕ^n denotes the nth repeated composition of ϕ with itself [11].

Example 2.1. Let (X, d) be a metric space, f and g self-maps on X, $\lambda > 0$ and ξ a nonnegative Lebesgue integrable self-map on $[0, \infty)$ for which $\int_0^{\delta} \xi(t)\, dt > 0$ for all $\delta > 0$. Define $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(s) = \int_0^s \xi(t)\, dt > 0$ and $\phi(s) = \lambda s$. Then, $\psi(d(gx, gy)) \leq \phi(\psi(M(x, y)))$ is equivalent to $\int_0^{d(gx,gy)} \xi(t)\, dt \leq \lambda \int_0^{M(x,y)} \xi(t)\, dt$. This shows that the (ψ, ϕ)-type contractive condition is a generalization of the contractive condition of integral type in [7]. Also if ϕ is a function that is continuous from the right and nondecreasing for which $\phi(t) < t$ for all $t > 0$, then $\psi(d(gx, gy)) \leq \phi(\psi(M(x, y)))$ is equivalent to $\int_0^{d(gx,gy)} \xi(t)\, dt \leq \phi(\int_0^{M(x,y)} \xi(t)\, dt)$. This shows that the (ψ, ϕ)-type contractive condition is a generalization of the generalized contractive condition of integral type in [8].

Example 2.2. Let (X, d) be metric space, $\lambda \geq 1$ and f and g self-maps on X. Define $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(s) = \lambda s$ and $\phi(s) = \ln(s + 1)$. Then, the relation $\psi(d(gx, gy)) \leq \phi(\psi(M(x, y)))$ is equivalent to $\lambda d(gx, gy) \leq \ln(M(x, y) + 1)$. If $\psi(s) = s^2 + s$ and $\phi(s) = \ln(s+1)$, then the relation $\psi(d(gx, gy)) \leq \phi(\psi(M(x, y)))$ is equivalent to $(d(gx, gy))^2 + d(gx, gy) \leq \ln(M(x, y) + 1)$. If $\psi(s) = s^2 + 1$ and $\phi(s) = \ln(\frac{s+1}{2})$, then the relation $\psi(d(gx, gy)) \leq \phi(\psi(M(x, y)))$ is equivalent to $(d(gx, gy))^2 \leq \ln(\frac{M(x,y)+1}{2}) - 1$.

These examples show that the class of (ψ, ϕ)-type contractive maps is bigger than the class of maps which satisfy the generalized contractive condition of integral type. Now, we are ready to state and prove our main results.

Lemma 2.1. *Let (X, d) be a metric space, $x_0 \in X$, f and g self-maps on X satisfying the (ψ, ϕ)-type contractive condition, $g(X) \subseteq f(X)$, $\delta(O(y_k, n)) > 0$ for all $k \geq 0$ and $n \geq 1$ and $\delta(O(y_0, \infty)) < \infty$. Then,*

$$\psi(\delta(O(y_k, n))) \leq \phi^k(\psi(\delta(O(y_0, \infty))))$$

for all $k \geq 1$.

Proof. For each $k \geq 0$ and $n \geq 1$ there exist integers i, j satisfying $k \leq i < j \leq k + n$ such that $\delta(O(y_k, n)) = d(y_i, y_j) = d(gx_i, gx_j)$. For such i, j we have

$$\begin{aligned}\psi(\delta(O(y_k, n))) &= \psi(d(y_i, y_j)) = \psi(d(gx_i, gx_j)) \leq \phi(\psi(M(x_i, x_j)))\\ &= \phi(\psi(\max\{d(fx_i, fx_j), d(fx_i, gx_i), d(fx_j, gx_j), d(fx_i, gx_j), d(fx_j, gx_i)\}))\\ &= \phi(\psi(\max\{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{j-1}, y_j), d(y_{i-1}, y_j), d(y_{j-1}, y_i)\}))\\ &\leq \phi(\psi(\delta(O(y_{i-1}, j - i + 1)))).\end{aligned}$$

Now, we claim that $\delta(O(y_k, n)) = d(y_k, y_j)$ for some integer j satisfying $k < j \leq k + n$. Otherwise, suppose that $\delta(O(y_k, n)) = d(y_i, y_j)$ with $i > k$. Thus, $i - 1 \geq k$ and so $O(y_{i-1}, j - i + 1) \subseteq O(y_k, n)$. Hence,

$$\psi(\delta(O(y_k, n))) \leq \phi(\psi(\delta(O(y_{i-1}, j - i + 1)))) \leq \phi(\psi(\delta(O(y_k, n)))) < \psi(\delta(O(y_k, n))).$$

This contradiction proves the claim. Since the function ϕ is nondecreasing, we have

$$\psi(\delta(O(y_k, n))) \leq \phi(\psi(\delta(O(y_{k-1}, j - i + 1)))) \leq \phi(\psi(\delta(O(y_{k-1}, n + 1)))).$$

Thus, we obtain

$$\begin{aligned}\psi(\delta(O(y_k, n))) &\leq \phi(\psi(\delta(O(y_{k-1}, n + 1)))) \leq \phi(\phi(\psi(\delta(O(y_{k-2}, n + 2)))))\\ &\leq \cdots \leq \phi^k(\psi(\delta(O(y_0, n + k)))).\end{aligned}$$

Therefore, the lemma follows because $\delta(O(y_0, \infty)) < \infty$ and the functions ϕ and ψ are nondecreasing. □

Theorem 2.2. *Let (X, d) be a complete metric space, f and g weakly compatible self-maps on X satisfying the (ψ, ϕ)-type contractive condition and $g(X) \subseteq f(X)$. Assume that $f(X)$ is a closed subset of X and that there exists $x_0 \in X$ such that $\delta(O(y_0, \infty)) < \infty$ and $\delta(O(y_k, n)) > 0$ for all $k \geq 0$ and $n \geq 1$. Then, f and g have a unique common fixed point.*

Proof. For all integers m and n with $m > n$ we have $d(y_n, y_m) \leq \delta(O(y_n, m))$. So,

$$\psi(d(y_n, y_m)) \leq \psi(\delta(O(y_n, m))) \leq \phi^n(\psi(\delta(O(y_0, m + n)))) \leq \phi^n(\psi(\delta(O(y_0, \infty)))).$$