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Coefficient estimates for a certain subclass of analytic and bi-univalent functions

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ABSTRACT

In this paper, we introduce and investigate an interesting subclass $\mathcal{H}_{\Sigma}^{h,p}$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{H}_{\Sigma}^{h,p}$, we obtain estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize and improve some recent work of Srivastava et al. [H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192].

1. Introduction and definitions

Let $\mathbb{R} = (-\infty, \infty)$ be the set of *real* numbers, $\mathbb{C}$ be the set of *complex* numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of *positive* integers. We let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of the *normalized* analytic function class $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also *univalent* in $\mathbb{U}$ (see, for details, [1,2]; see also some of the recent investigations [3–8], dealing with various interesting subclasses of the analytic function class $\mathcal{A}$ and the univalent function class $\mathcal{A}$).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U})$$

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and

$$f^{-1}\big(f(w)\big) = w \quad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor–Maclaurin series expansion (1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\ \log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ.

Lewin [9] first investigated the bi-univalent function class Σ and showed that

$$|a_2| < 1.51.$$

Subsequently, Brannan and Clunie [10] conjectured that

$$|a_2| \leqq \sqrt{2}.$$

Netanyahu [11], on the other hand, showed that

$$\max_{f\in\Sigma} |a_2| = \frac{4}{3}.$$

The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\})$$

is presumably still an open problem. In [12] (see also [13] and [14]), certain subclasses of the bi-univalent function class Σ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found.

Recently, Srivastava et al. [15] introduced the following subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$.

Definition 1 (*See [15]*)**.** A function $f(z)$ given by the Taylor–Maclaurin series expansion (1) is said to be in the class $\mathcal{H}_\Sigma^\alpha$ $(0 < \alpha \leqq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\big(f'(z)\big)\right| \leqq \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \tag{2}$$

and

$$\left|\arg\big(g'(w)\big)\right| \leqq \frac{\alpha\pi}{2} \quad (w \in \mathbb{U};\ 0 < \alpha \leqq 1), \tag{3}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{4}$$

Theorem 1 (*See [15]*)**.** *Let the function $f(z)$ given by (1) be in the bi-univalent function class $\mathcal{H}_\Sigma^\alpha$ $(0 < \alpha \leqq 1)$. Then*

$$|a_2| \leqq \alpha\sqrt{\frac{2}{\alpha+2}} \quad \textit{and} \quad |a_3| \leqq \frac{\alpha(3\alpha+2)}{3}. \tag{5}$$

Definition 2 (*See [15]*)**.** A function $f(z)$ given by the Taylor–Maclaurin series expansion (1) is said to be in the class $\mathcal{H}_\Sigma^\beta$ $(0 \leqq \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\big(f'(z)\big) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1) \tag{6}$$

and

$$\Re\big(g'(w)\big) > \beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1), \tag{7}$$

where the function g is defined by (4).