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Basins of attraction of equilibrium points of second order difference equations

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ABSTRACT

We investigate the basins of attraction of equilibrium points and period-two solutions of the difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots,$$

where f is decreasing in the first and increasing in the second variable. We show that the boundaries of the basins of attraction of different locally asymptotically stable equilibrium points are in fact the global stable manifolds of neighboring saddle or non-hyperbolic equilibrium points.

1. Introduction and preliminaries

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$. Let $\bar{x} \in I$, be an equilibrium point of a difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, n = 0, 1, \ldots, \tag{1}$$

where f is a continuous function and is decreasing in the first and increasing in the second variable. There are several global attractivity results for Eq. (1) which give the sufficient conditions for all solutions to approach a unique equilibrium. These results were used efficiently in monograph [1] to study the global behavior of solutions of second order linear fractional difference equations with non-negative initial conditions and parameters. One such result is the following theorem.

Theorem 1 (*See [1–4]*)**.** *Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:*
(a) *$f(x, y)$ is non-increasing in the first and non-decreasing in the second variable.*
(b) *Eq. (1) has no minimal period-two solutions in $[a, b]$.*
Then every solution of Eq. (1) converges to $\bar{x}$.

Another related important result is the following theorem.

Theorem 2 (*See [5]*)**.** *Let $I \subseteq R$ be an interval and let $f \in C[I \times I, I]$ be a function which is non-increasing in the first and non-decreasing in the second variable. Then for every solution of Eq. (1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*
(i) *Eventually they are both monotonically increasing.*
(ii) *Eventually they are both monotonically decreasing.*
(iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

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Theorem 1 is a special case of Theorem 2 if we additionally assume the non-existence of period-two solutions and the uniqueness of the equilibrium. None of these results provide any information about basins of attraction of different equilibrium points when there exist several equilibrium points, as in the case of Eq. (6). In this paper, by using theory of monotone maps in the plane, we give precise description of basins of attraction of different equilibrium points. Similar results have been obtained in [6] for Eq. (1) when f is increasing in both arguments.

We now give some basic notions about monotone maps in the plane.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are said to be *related* if $\mathbf{v} \preceq \mathbf{w}$ or $\mathbf{w} \preceq \mathbf{v}$. Also, a strict inequality between points may be defined as $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v} \preceq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. A stronger inequality may be defined as $\mathbf{v} = (v_1, v_2) \ll \mathbf{w} = (w_1, w_2)$ if $\mathbf{v} \preceq \mathbf{w}$ with $v_1 \neq w_1$ and $v_2 \neq w_2$. For $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$, the *order interval* $[\![\mathbf{u}, \mathbf{v}]\!]$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{u} \preceq \mathbf{x} \preceq \mathbf{v}$. The interior of a set A is denoted as *intA*.

A map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{S} \to \mathcal{S}$. The map T is *monotone* if $\mathbf{v} \preceq \mathbf{w}$ implies $T(\mathbf{v}) \preceq T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$, and it is *strongly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \ll T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. The map is *strictly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \prec T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. Clearly, being related is invariant under the iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering is defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

If T is a differentiable map on a nonempty set $\mathcal{S}$, a sufficient condition for T to be strongly monotone with respect to the SE ordering is that the Jacobian matrix at all points $\mathbf{x}$ has the sign configuration

$$\operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \tag{2}$$

provided that $\mathcal{S}$ is open and convex.

For $\mathbf{x} \in \mathbb{R}^2$, define $\mathcal{Q}_\ell(\mathbf{x})$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at $\mathbf{x}$ and numbered in a counterclockwise direction, for example, $Q_1(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 : x_1 \leq y_1, x_2 \leq y_2\}$. The (open) ball of radius r centered at $\mathbf{x}$ is denoted with $\mathcal{B}(\mathbf{x}, r)$. If $\mathcal{K} \subset \mathbb{R}^2$ and $r > 0$, write $\mathcal{K} + \mathcal{B}(\mathbf{0}, r) := \{\mathbf{x} : \mathbf{x} = \mathbf{k} + \mathbf{y}$ for some $\mathbf{k} \in \mathcal{K}$ and $\mathbf{y} \in \mathcal{B}(\mathbf{0}, r)\}$. If $\mathbf{x} \in [-\infty, \infty]^2$ is such that $\mathbf{x} \preceq \mathbf{y}$ for every $\mathbf{y}$ in a set $\mathcal{Y}$, we write $\mathbf{x} \preceq \mathcal{Y}$. The inequality $\mathcal{Y} \preceq \mathbf{x}$ is defined similarly. The basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p [7]. The next five results, from [8,9], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by Smith in [10–12].

Theorem 3. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \operatorname{int}(\mathcal{Q}_1(\overline{\mathrm{x}}) \cup \mathcal{Q}_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. *The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.*

b. *The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 5 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 4. *For the curve $\mathcal{C}$ of Theorem 3 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. *The map T has no fixed points nor periodic points of minimal period two in Δ.*

ii. *The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(\mathrm{x}) = \overline{\mathrm{x}}$ has no solutions $\mathrm{x} \in \Delta$.*

iii. *The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(\mathrm{x}) = \overline{\mathrm{x}}$ has no solutions $\mathrm{x} \in \Delta$.*

Corollary 1 ([8]). *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^2$, and if T has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

For maps that are strongly competitive near the fixed point, hypothesis b of Theorem 3 reduces just to $|\lambda| < 1$. This follows from a change of variables [13] that allows the Perron–Frobenius Theorem to be applied. Also, one can show that in such a case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.